

# FORCING TWO-POINT HERMITE DATA: THE FULL FAMILY OF SPLITS

DAVID PAN

ABSTRACT. Eighth in a series descending from [4]: we unify the one-sided theory of [5] and the balanced theory of [11] into a single family. For the class  $f^{(n)} \geq 0$  on an interval with two-point Hermite data split  $(p, q)$ — $p$  derivatives prescribed at the left endpoint,  $q$  at the right,  $p + q = n$ —classical disconjugacy gives the Green function the sign  $(-1)^q$ , so *every* split carries a forcing theory: the Hermite interpolant is a pointwise-greatest element for  $q$  odd and least for  $q$  even, and forcing reduces to that one barrier. The geometry is split-blind: for every  $(p, q)$  the forcing set is a union of two convex charts, each linearly isomorphic, modulo the line of constants, to the cone of polynomials of degree  $\leq n - 2$  nonnegative on  $[0, 1]$ . So each chart is semidefinite-representable for every  $n$  and polyhedral exactly when  $n \leq 3$ , while the forcing set itself is convex only at  $n = 2$ , is a nonconvex union of two polyhedral cones at  $n = 3$ , and for  $n \geq 4$  is not even a finite union of polyhedral cones: the order decides everything, the split nothing. One sign criterion,  $(-1)^{q-k} \partial_\nu^k u \geq 0$  at both endpoints, contains the descending criterion of [5] (the split  $(n - 1, 1)$ ) and the alternating criterion of [11] (the balanced split), echoing the sign patterns of [9, 10]. The Hermite cardinals are binomial tails,  $H_{k0}(x) = \frac{x^k}{k!} \mathbb{P}(\text{Bin}(n - 1 - k, x) \leq p - 1 - k)$ , and the worst-case overshoot under unit data bounds is the maximum of one universal extremal  $S^{(p,q)}$ , which at  $(n - 1, 1)$  is exactly the truncated-exponential family  $P_n$  of [5]. The main theorem is the ceiling: if  $p_\nu/n_\nu \rightarrow \lambda$ , then  $E^{(p_\nu, q_\nu)} \rightarrow e^{\max(\lambda, 1 - \lambda)} - 1$ . This proves the exponent conjecture of [11, Problem 8.3], makes the balanced ceiling  $\sqrt{e} - 1$  of [11, Theorem 7.2] unconditional, and recovers  $E(n) \rightarrow e - 1$  of [5]. The proof is two binomial-tail estimates: a convex-combination bound through the partition of unity, and Hoeffding’s inequality. Along fixed-ratio sequences the balanced split minimizes the limiting ceiling—asymptotically, sharing the data equally between the endpoints is the best way to control the class—and at each finite order the near-balanced splits tie the balanced one exactly:  $S^{(p, p+1)} \equiv S^{(p+1, p+1)}$ , by a Pascal-step identity of the binomial tails.

## 1. INTRODUCTION

Two installments of this series solved two instances of the same problem. In [5], the class  $f^{(n)} \geq 0$  carried  $n - 1$  conditions at the left endpoint and one at the right—the split  $(n - 1, 1)$ —and the theory produced the descending criterion, a union of two cones, and an overshoot that climbs to  $e - 1$ . In [11], the same class carried the balanced split  $(m, m)$  of  $n = 2m$ , and produced an alternating criterion, the same two-chart geometry, and an overshoot whose ceiling appeared to be  $\sqrt{e} - 1$ —proved there only up to a centrality conjecture, and recorded as an exponent conjecture for general splits in [11, Problem 8.3].

This paper does the whole family at once, and settles that conjecture. Fix  $n \geq 2$  and a split  $(p, q)$ ,  $p, q \geq 1$ ,  $p + q = n$ : prescribe  $f^{(k)}(0) = \alpha_k$  for  $0 \leq k \leq p - 1$  and  $f^{(k)}(1) = \beta_k$  for  $0 \leq k \leq q - 1$ . Classical disconjugacy theory [1, Ch. III, §5] gives the two-point Green function of  $f^{(n)}$  with these boundary conditions the sign  $(-1)^q$ , uniformly in the split; everything else

---

*Date:* August 9, 2026.

*2020 Mathematics Subject Classification.* Primary 26A51, 26D10; Secondary 34B27, 41A05, 60F10.

*Key words and phrases.* maximum principle, Hermite interpolation, disconjugacy, Green’s function, overshoot, binomial tail bounds.

follows from that one sign and from the structure of the two-point Hermite basis. Four points summarize the paper.

- **Every split has a forcing theory, and its geometry is split-blind** (Theorems 2.1, 3.1). For  $q$  odd the Hermite interpolant is the pointwise-greatest member and the theory is about the maximum principle; for  $q$  even, least and minimum. In all cases the forcing set is a union of two charts, each isomorphic modulo constants to the cone  $\mathcal{P}_{n-2}$  of polynomials of degree  $\leq n-2$  nonnegative on  $[0, 1]$ . In particular each chart is polyhedral iff  $n \leq 3$ ; the forcing set is nonconvex for every  $n \geq 3$  and, for  $n \geq 4$ , is not a finite union of polyhedral cones, *regardless of the split*: the appearance of curvature at differential order four, observed at  $(3, 1)$  in [5] and at  $(2, 2)$  in [11], is a theorem about the order alone.
- **One sign criterion governs the family** (Theorem 4.3):  $(-1)^{q-k} \partial_\nu^k u \geq 0$  at both endpoints forces, for arbitrary endpoint values. At  $q = 1$  this *is* the descending criterion of [5, Theorem 4.1]; at  $(m, m)$  it is the alternating criterion of [11, Theorem 4.2]; the sign patterns of [9, Theorem 3.1] and [10, Theorem 4.7] are the same pattern at the PDE level.
- **The cardinals are binomial tails** (Lemma 4.2):  $H_{k0}(x) = \frac{x^k}{k!} \mathbb{P}(\text{Bin}(n-1-k, x) \leq p-1-k)$ . The partition of unity becomes complementarity of binomial events, the matchbox identity of [11] becomes the symmetry of  $\text{Bin}(2m-1, \frac{1}{2})$ , and the overshoot analysis reduces to the law of large numbers with Hoeffding's quantitative tail [2].
- **The ceiling theorem** (Theorem 6.2). The worst-case overshoot under unit data bounds is  $E^{(p,q)} = \max_{[0,1]} S^{(p,q)}$  for one universal extremal per split—equal at  $(n-1, 1)$  to the family  $P_n$  of [5, Theorem 6.8], exactly—and along any sequence of splits with  $p_\nu/n_\nu \rightarrow \lambda$ ,

$$E^{(p_\nu, q_\nu)} \rightarrow e^{\max(\lambda, 1-\lambda)} - 1.$$

This proves the conjecture of [11, Problem 8.3]; at  $\lambda = \frac{1}{2}$  it upgrades the balanced ceiling  $\sqrt{e} - 1$  of [11, Theorem 7.2] from conditional to unconditional (the centrality conjecture there remains open, but now governs only exact finite-order values); at  $\lambda = 1$  it recovers  $E(n) \rightarrow e - 1$  [5, Theorem 6.8] by a new route. The exponent  $\max(\lambda, 1-\lambda)$  is minimized at the balanced split: asymptotically, distributing the data evenly is optimal—and at each finite order the near-balanced splits tie the balanced one exactly (Proposition 5.2).

Section 2 sets up the reduction; Section 3 the split-blind geometry; Section 4 the cardinal identities and the sign criterion; Section 5 the overshoot; Section 6 the ceiling theorem; Section 7 problems. All exact claims are verified in rational arithmetic; the script accompanies the source.

## 2. SETTING AND THE REDUCTION

Work on  $[0, 1]$  (general intervals by affine substitution, as in [11, Remark 5.3]). Fix the split  $(p, q)$  and data  $D = (\alpha; \beta) \in \mathbb{R}^n$ , and let

$$\mathcal{K}(D) = \{u \in C^n([0, 1]) : u^{(n)} \geq 0, u^{(k)}(0) = \alpha_k \ (k < p), u^{(k)}(1) = \beta_k \ (k < q)\}.$$

Since  $p, q \geq 1$ , both endpoint values are prescribed;  $D$  is *max-forcing* if every member satisfies  $u \leq \max\{\alpha_0, \beta_0\}$ , *min-forcing* if every member satisfies  $u \geq \min\{\alpha_0, \beta_0\}$ , and, as throughout the series, these notions are discussed only for nonempty classes. Let  $v_D$  be the  $(p, q)$  Hermite interpolant, the unique polynomial of degree  $\leq n-1$  with the data (two-point Hermite unisolvence). Write  $\partial_\nu$  for the outward derivative:  $\partial_\nu^k u(0) = (-1)^k u^{(k)}(0)$ ,  $\partial_\nu^k u(1) = u^{(k)}(1)$ .

Two classical facts about the boundary value problem  $u^{(n)} = f$  with zero  $(p, q)$  data [1, Ch. III, §5]: it has a Green function  $G = G_{p,q}$ , continuous on the square, with

$$(-1)^q G \geq 0, \quad G \not\equiv 0;$$

and every  $h \in C^n([0, 1])$  with zero  $(p, q)$  data satisfies  $h(x) = \int_0^1 G(x, s) h^{(n)}(s) ds$  (the integral solves the same problem with the same right-hand side, and the difference is a polynomial of degree  $\leq n - 1$  with zero Hermite data, hence zero).

**Theorem 2.1** (Reduction, all splits). *Let  $q$  be odd. Then  $v_D$  is the pointwise-greatest element of  $\mathcal{K}(D)$ ;  $D$  is max-forcing iff  $v_D \leq \max\{\alpha_0, \beta_0\}$  on  $[0, 1]$ ; and no data of this split is min-forcing. For  $q$  even, the mirror statements hold: least element, min-forcing iff  $v_D \geq \min\{\alpha_0, \beta_0\}$ , and no data is max-forcing.*

*Proof.* For  $u \in \mathcal{K}(D)$ ,  $h = u - v_D$  has zero  $(p, q)$  data, so  $(-1)^q h = \int (-1)^q G u^{(n)} \geq 0$ : for  $q$  odd every member lies below  $v_D$ , for  $q$  even above. The barrier criterion follows since  $v_D$  is itself a member. For the no-go halves: pick interior points with  $G(x_0, s_0) \neq 0$  (possible since  $G$  is continuous and not identically zero, and  $G$  vanishes when  $x$  is an endpoint), a bump  $f \in C_c^\infty$  concentrated near  $s_0$  with  $f \geq 0$ , and set  $w = \int G(\cdot, s) f(s) ds$ : then  $w$  has zero data,  $w^{(n)} = f \geq 0$ , and  $(-1)^q w(x_0) > 0$ . For any  $u_0 \in \mathcal{K}(D)$ , the family  $u_0 + t w$  ( $t \geq 0$ ) stays in the class and is unbounded at  $x_0$  in the direction that the opposite principle would need to control.  $\square$

The involution  $\rho u(x) := (-1)^n u(1 - x)$  preserves the class  $((\rho u)^{(n)}(x) = u^{(n)}(1 - x) \geq 0)$  and exchanges the splits  $(p, q) \leftrightarrow (q, p)$ , with data  $\rho(\alpha; \beta) = (((-1)^{n+k} \beta_k); ((-1)^{n+k} \alpha_k))$ ; for  $n$  even it preserves max/min-forcing, for  $n$  odd it exchanges them (consistently with the parity of Theorem 2.1:  $q$  and  $p = n - q$  have equal parity iff  $n$  is even). So the  $(q, p)$  theory is the mirror of the  $(p, q)$  theory, and  $E^{(p,q)} = E^{(q,p)}$  below.

### 3. THE SPLIT-BLIND GEOMETRY

**Theorem 3.1** (Structure). *Fix a split  $(p, q)$  with  $q$  odd (the  $q$ -even case mirrors with min in place of max and reversed inequalities). Define the charts*

$$\mathcal{R} = \{D : v_D \leq \beta_0 \text{ on } [0, 1]\}, \quad \mathcal{L} = \{D : v_D \leq \alpha_0 \text{ on } [0, 1]\}.$$

*Then:*

- (i) *the forcing set is  $\mathcal{R} \cup \mathcal{L}$ , with  $\mathcal{R} \subseteq \{\beta_0 \geq \alpha_0\}$  and  $\mathcal{L} \subseteq \{\alpha_0 \geq \beta_0\}$  automatically;*
- (ii)  *$D \mapsto \chi_D := (v_D - \beta_0)/(x - 1)$  is linear onto the polynomials of degree  $\leq n - 2$  with kernel the line of constants, and  $\mathcal{R} = \{\chi_D \geq 0 \text{ on } [0, 1]\}$ ; likewise  $\mathcal{L} = \{(v_D - \alpha_0)/x \leq 0\}$ . Modulo the constants line, each chart is linearly isomorphic to  $\pm \mathcal{P}_{n-2}$ , the cone of polynomials of degree  $\leq n - 2$  nonnegative on  $[0, 1]$ ;*
- (iii) *each chart is a closed convex cone, semidefinite-representable via Markov–Lukács [12, §1.21] for every  $n$ , and polyhedral iff  $n \leq 3$ . The forcing set itself is all of  $\mathbb{R}^2$  for  $n = 2$ ; a nonconvex union of two polyhedral cones for  $n = 3$ ; and for  $n \geq 4$  it is nonconvex and not even a finite union of polyhedral cones.*

*In particular the forcing geometry depends only on the order  $n$ , not on the split.*

*Proof.* (i) is Theorem 2.1 split by which endpoint value is the larger; evaluating the chart condition at the other endpoint gives the half-space automatically. (ii)  $v_D - \beta_0$  vanishes at 1, so  $\chi_D$  is a polynomial of degree  $\leq n - 2$  depending linearly on  $D$ ;  $(x - 1) \leq 0$  on  $[0, 1]$  converts  $v_D - \beta_0 \leq 0$  into  $\chi_D \geq 0$ ; the kernel of  $D \mapsto \chi_D$  is  $\{v_D \equiv \beta_0\}$ , the constant data, and the dimensions  $n$  and  $n - 1$  make the map onto. (iii) Preimages of closed convex sets under continuous linear maps are closed convex, and the semidefinite representation composes with the linear map; the quotient by the constants line is a linear isomorphism onto  $\mathcal{P}_{n-2}$ . Polyhedrality:  $\mathcal{P}_{n-2}$  is a half-line for  $n = 2$ , a two-facet sector for  $n = 3$ , and for  $n \geq 4$  its quadratic section carries the continuum of extreme rays  $(x - s)^2$  [3], so the charts are polyhedral exactly for  $n \leq 3$ .

For  $n = 2$  the barrier is affine, so every data forces and the forcing set is all of  $\mathbb{R}^2$ . For  $n \geq 4$ : if the forcing set were a finite union of polyhedra, so would be its intersection with the half-space  $\{\beta_0 \geq \alpha_0\}$ , which by (i) is the chart  $\mathcal{R}$ ; a convex set that is a finite union of polyhedra is itself a polyhedron, contradicting the section argument.

Nonconvexity for  $n \geq 3$ : for the splits  $(n-1, 1)$  and  $(1, n-1)$  this is [5, Theorem 5.2] and its  $\rho$ -image. For  $p, q \geq 2$ , note the chart anchors  $\chi_{\mathcal{R}}(1) = v'_D(1) = \beta_1$  and, on the left chart,  $(v_D - \alpha_0)'(0) = \alpha_1$ , giving the necessary conditions  $\beta_1 \geq 0$  on  $\mathcal{R}$  and  $\alpha_1 \leq 0$  on  $\mathcal{L}$ . The data  $a$  (all zero except  $\alpha_1 = 1, \beta_0 = 1$ ) forces: its barrier is  $-H_{00} + H_{10} \leq 0$  since  $H_{10} \leq H_{00}$  termwise in the binomial forms of Lemma 4.1. The data  $b$  (all zero except  $\alpha_0 = 1, \beta_1 = -1$ ) forces: its barrier relative to  $\alpha_0$  is  $-H_{01} - H_{11} = -[H_{00}^{(q,p)} - H_{10}^{(q,p)}](1-x) \leq 0$  by the same inequality for the reversed split. Their midpoint has  $\alpha_0 = \beta_0, \alpha_1 = \frac{1}{2} > 0$ , and  $\beta_1 = -\frac{1}{2} < 0$ : at the benchmark tie it violates the necessary conditions of both charts. (For  $q$  even the mirrored witnesses of the same shape work; they are listed in the verification script.)  $\square$

*Remark 3.2.* The chart identity explains a coincidence across the series: [5, §2] observed that its cone is the preimage of a nonnegativity cone, and [11, Theorem 3.1] proved the balanced case; Theorem 3.1 shows the mechanism is universal. What distinguishes the splits is not the geometry of one chart but the *data map* onto it—and, downstream, the sign criteria and overshoot constants below.

#### 4. CARDINALS AS BINOMIAL TAILS; THE SIGN CRITERION

Let  $H_{k0}, H_{k1}$  be the  $(p, q)$  Hermite cardinal polynomials ( $H_{k0}^{(j)}(0) = \delta_{jk}$  for  $j < p$ , zero data at 1; mirrored for  $H_{k1}$ ).

**Lemma 4.1** (Binomial form). *For  $0 \leq k \leq p-1$ ,*

$$H_{k0}(x) = \frac{x^k}{k!} (1-x)^q \sum_{j=0}^{p-1-k} \binom{q-1+j}{j} x^j, \quad H_{k1}(x) = (-1)^k H_{k0}^{(q,p)}(1-x),$$

*so every cardinal is one-signed on  $[0, 1]$ , and  $H_{00} + H_{01} \equiv 1$ .*

*Proof.* As in [11, Lemma 4.1] with  $(m, m)$  replaced by  $(p, q)$ : the sum is the degree- $(p-1-k)$  truncation of the binomial series  $(1-x)^{-q}$ , so the displayed function is  $\frac{x^k}{k!} + O(x^p)$  at 0 and vanishes to order  $q$  at 1; degree and unisolvence do the rest. The reflection identity is the chain rule, and  $H_{00} + H_{01}$  interpolates 1.  $\square$

**Lemma 4.2** (Binomial tails). *For  $0 \leq k \leq p-1$ ,*

$$H_{k0}(x) = \frac{x^k}{k!} \mathbb{P}(\text{Bin}(n-1-k, x) \leq p-1-k);$$

*in particular  $H_{00}(x) = \mathbb{P}(\text{Bin}(n-1, x) \leq p-1)$  and  $H_{01}(x) = \mathbb{P}(\text{Bin}(n-1, x) \geq p)$ .*

*Proof.* In a sequence of independent coin flips with head probability  $x$ , the truncated series in Lemma 4.1 (with  $k$  absorbed) is the probability that the  $q$ -th tail arrives with at most  $p-1-k$  heads before it, i.e. within the first  $n-1-k$  flips; and “the  $q$ -th tail occurs within  $N$  flips” is the event “at most  $N-q$  heads among  $N$  flips” with  $N = n-1-k$ , giving the binomial form. The identity is between polynomials in  $x$ , so it holds identically (it is also verified exactly, coefficient by coefficient, in the script). For  $k=0$  the two displayed forms are complementary events, reproving the partition of unity; at the balanced split and  $x = \frac{1}{2}$ , the symmetry  $\mathbb{P}(\text{Bin}(2m-1, \frac{1}{2}) \geq m) = \frac{1}{2}$  is the matchbox identity of [11, Theorem 7.2].  $\square$

**Theorem 4.3** (The sign criterion of the family). *If*

$$(-1)^{q-k} \partial_\nu^k u \geq 0 \quad \text{at both endpoints} \quad (1 \leq k \leq p-1 \text{ at } 0, \quad 1 \leq k \leq q-1 \text{ at } 1),$$

*then  $D$  forces (max- for  $q$  odd, min- for  $q$  even), for arbitrary endpoint values  $\alpha_0, \beta_0$ .*

*Proof.* Let  $q$  be odd and suppose  $\beta_0 \geq \alpha_0$  (the case  $\alpha_0 > \beta_0$  uses the left barrier identically). Using  $H_{00} + H_{01} = 1$ ,

$$v_D - \beta_0 = \sum_{k \geq 1} \alpha_k H_{k0} + (\alpha_0 - \beta_0) H_{00} + \sum_{k \geq 1} \beta_k H_{k1}.$$

The hypothesis translates to  $\alpha_k \leq 0$  and  $(-1)^k \beta_k \leq 0$  (for  $q$  odd), so by Lemma 4.1 every term is  $\leq 0$ :  $v_D \leq \beta_0 = \max\{\alpha_0, \beta_0\}$ , and Theorem 2.1 concludes. For  $q$  even all signs flip.  $\square$

**Corollary 4.4.** *At  $q = 1$  the criterion reads  $f^{(k)}(0) \leq 0$  for  $1 \leq k \leq n-2$ : the descending criterion of [5, Theorem 4.1]. At  $(m, m)$  it is the alternating criterion of [11, Theorem 4.2]. The descending and alternating criteria of this series are one theorem, read at two splits.*

## 5. THE OVERSHOOT

Define, for  $q$  odd,

$$E^{(p,q)} = \sup \left\{ \max_{[0,1]} u - \max\{\alpha_0, \beta_0\} : u \in \mathcal{K}(D), |\alpha_k| \leq 1, |\beta_k| \leq 1 \ (k \geq 1) \right\},$$

values unconstrained, and for  $q$  even the mirror quantity  $\sup(\min\{\alpha_0, \beta_0\} - \min u)$ .

**Theorem 5.1** (Overshoot). *For every split,*

$$E^{(p,q)} = \max_{[0,1]} S^{(p,q)}, \quad S^{(p,q)}(x) := \sum_{k=1}^{p-1} H_{k0}(x) + \sum_{k=1}^{q-1} H_{k0}^{(q,p)}(1-x),$$

*and  $(-1)^{q+1} S^{(p,q)}$  is the barrier of the universal data  $\alpha_k = (-1)^{q+1}$ ,  $\beta_k = (-1)^{q+k+1}$ ,  $\alpha_0 = \beta_0 = 0$ : overshoot  $+S$  for  $q$  odd, sag  $-S$  for  $q$  even. Moreover  $E^{(p,q)} = E^{(q,p)}$ , and*

$$S^{(n-1,1)}(x) = \sum_{k=1}^{n-2} \frac{x^k - x^{n-1}}{k!} = P_n(x),$$

*the extremal family of [5, Theorem 6.8], exactly.*

*Proof.* The bound and attainment are as in [11, Theorem 7.1], with the value part controlled by  $\alpha_0 H_{00} + \beta_0 H_{01} \leq \max\{\alpha_0, \beta_0\}$  (positivity and partition of unity) and  $|H_{k1}(x)| = H_{k0}^{(q,p)}(1-x)$ ; the stated data produce  $(-1)^{q+1} S$  identically, a member of its own class with benchmark 0. Symmetry under  $\rho$  gives  $E^{(p,q)} = E^{(q,p)}$ . For  $(n-1, 1)$ : by Lemma 4.1 with  $q = 1$  the inner sum telescopes,

$$H_{k0}(x) = \frac{x^k}{k!} (1-x) \sum_{j=0}^{n-2-k} x^j = \frac{x^k - x^{n-1}}{k!},$$

and summing over  $1 \leq k \leq n-2$  gives  $P_n$ .  $\square$

**Proposition 5.2** (Near-balanced ties). *For every  $p \geq 1$ ,  $S^{(p,p+1)} \equiv S^{(p+1,p+1)}$  identically on  $[0, 1]$ ; hence*

$$E^{(p,p+1)} = E^{(p+1,p)} = E^{(p+1,p+1)};$$

*at odd order  $n = 2p + 1$ , both central splits have exactly the worst case of the balanced split of order  $2p + 2$ .*

*Proof.* Conditioning a binomial on its last trial gives the Pascal step

$$\mathbb{P}(\text{Bin}(N+1, x) \leq t) = \mathbb{P}(\text{Bin}(N, x) \leq t-1) + (1-x) \mathbb{P}(\text{Bin}(N, x) = t).$$

Apply it through Lemma 4.2 with  $N = 2p - k$ ,  $t = p - k$  to compare cardinals of the two splits (write  $y = 1 - x$  and  $X = x^p(1-x)^{p+1}$ ):

$$H_{k0}^{(p,p+1)}(x) - H_{k0}^{(p+1,p+1)}(x) = -\frac{x^k}{k!}(1-x) \mathbb{P}(\text{Bin}(2p-k, x) = p-k) = -c_k X,$$

$$H_{k0}^{(p+1,p)}(y) - H_{k0}^{(p+1,p+1)}(y) = \frac{y^k}{k!} y \mathbb{P}(\text{Bin}(2p-k, y) = p-k) = +c_k X,$$

with  $c_k = \binom{2p-k}{p-k}/k!$ . Summing the first line over  $1 \leq k \leq p-1$  and the second over  $1 \leq k \leq p$ , and subtracting the one extra left cardinal of the balanced split,  $H_{p0}^{(p+1,p+1)}(x) = X/p!$ , gives

$$S^{(p,p+1)} - S^{(p+1,p+1)} = X \left( \sum_{k \leq p} c_k - \sum_{k \leq p-1} c_k - \frac{1}{p!} \right) = X \left( c_p - \frac{1}{p!} \right) = 0,$$

since  $c_p = \binom{p}{0}/p! = 1/p!$ .  $\square$

## 6. THE CEILING THEOREM

**Lemma 6.1** (Two tail bounds). *For every split and every  $x \in [0, 1]$ :*

- (a)  $S^{(p,q)}(x) \leq (e^x - 1)H_{00}(x) + (e^{1-x} - 1)H_{01}(x) \leq e^{\max(x, 1-x)} - 1$ , a convex combination; in particular  $E^{(p,q)} < e - 1$  for every split.
- (b) (Hoeffding [2]) For  $t > 0$ :  $H_{01}(x) = \mathbb{P}(\text{Bin}(n-1, x) \geq p) \leq \exp(-2(n-1)t^2)$  whenever  $x \leq \frac{p}{n-1} - t$ ; symmetrically  $H_{00}(x) \leq \exp(-2(n-1)t^2)$  whenever  $x \geq \frac{p-1}{n-1} + t$ ; and for  $k \geq 1$ ,

$$0 \leq \frac{x^k}{k!} - H_{k0}(x) = \frac{x^k}{k!} \mathbb{P}(\text{Bin}(n-1-k, x) \geq p-k) \leq \frac{x^k}{k!} e^{-2(n-1-k)t^2}$$

$$\text{whenever } x \leq \frac{p-k}{n-1-k} - t.$$

*Proof.* (a) Termwise, the truncated binomial sum for  $H_{k0}$  is dominated by the one for  $H_{00}$ , so  $H_{k0} \leq \frac{x^k}{k!} H_{00}$  and, mirrored,  $H_{k0}^{(q,p)}(1-x) \leq \frac{(1-x)^k}{k!} H_{01}(x)$ ; summing the exponential series and using  $H_{00} + H_{01} = 1$  gives the convex combination, which is at most the larger of the two values. The strict bound  $E < e - 1$  follows as in [11, Theorem 7.2]:  $S$  is continuous, vanishes at the endpoints, and is strictly below  $e - 1$  in the interior. (b) is Hoeffding's inequality applied to the binomial tails of Lemma 4.2, whose thresholds sit at relative deviation  $t$  from the mean.  $\square$

**Theorem 6.2** (The ceiling). *Let  $(p_\nu, q_\nu)$  be splits with  $n_\nu = p_\nu + q_\nu \rightarrow \infty$  and  $p_\nu/n_\nu \rightarrow \lambda \in [0, 1]$ . Then*

$$E^{(p_\nu, q_\nu)} \rightarrow e^{\max(\lambda, 1-\lambda)} - 1.$$

*Proof.* Write  $M = \max(\lambda, 1-\lambda)$  and  $S_\nu = S^{(p_\nu, q_\nu)}$ , and fix  $\varepsilon \in (0, \frac{1}{2})$ .

*Upper bound.* For large  $\nu$ , split  $[0, 1]$  into three regions. On  $|x - \lambda| \leq \varepsilon$ : by Lemma 6.1(a),  $S_\nu(x) \leq e^{\max(x, 1-x)} - 1 \leq e^{M+\varepsilon} - 1$ . On  $x \leq \lambda - \varepsilon$ : since  $p_\nu/(n_\nu - 1) \rightarrow \lambda$ , eventually  $x \leq \frac{p_\nu}{n_\nu - 1} - \frac{\varepsilon}{2}$  throughout the region, so by (a) and (b),

$$S_\nu(x) \leq (e^x - 1) + (e - 1)H_{01}(x) \leq (e^\lambda - 1) + (e - 1)e^{-(n_\nu - 1)\varepsilon^2/2},$$

which is  $\leq e^M - 1 + o(1)$ . On  $x \geq \lambda + \varepsilon$ , symmetrically,  $S_\nu(x) \leq (e^{1-\lambda-\varepsilon} - 1) + (e - 1)e^{-(n_\nu - 1)\varepsilon^2/2} \leq e^M - 1 + o(1)$ . Hence  $\limsup_\nu \max S_\nu \leq e^{M+\varepsilon} - 1$  for every  $\varepsilon$ , so  $\leq e^M - 1$ .

*Lower bound.* Choose the endpoint whose data fraction attains  $M$ ; say  $1 - \lambda = M$  (else exchange the roles by  $\rho$ ), so  $q_\nu/n_\nu \rightarrow M > 0$  and  $q_\nu \rightarrow \infty$ . Fix  $x_0 \in (\lambda, 1)$  and an integer  $K$ .

For each fixed  $k \leq K$  and large  $\nu$ , the mirrored deficiency bound of Lemma 6.1(b) (for the split  $(q_\nu, p_\nu)$  at the point  $1 - x_0 < 1 - \lambda$ ) gives

$$H_{k0}^{(q_\nu, p_\nu)}(1 - x_0) \longrightarrow \frac{(1 - x_0)^k}{k!},$$

so  $\liminf_\nu S_\nu(x_0) \geq \sum_{k=1}^K \frac{(1-x_0)^k}{k!}$ ; letting  $K \rightarrow \infty$  and then  $x_0 \downarrow \lambda$ ,

$$\liminf_\nu \max_{[0,1]} S_\nu \geq e^{1-x_0} - 1 \longrightarrow e^M - 1. \quad \square$$

**Corollary 6.3.**  *$E_m \rightarrow \sqrt{e} - 1$  for the balanced splits, unconditionally: the conditionality of [11, Theorem 7.2] is removed. (The centrality conjecture [11, Problem 8.2] remains open, but now concerns only the exact values  $E_m = S_m(\frac{1}{2})$  at finite  $m$ .) At  $\lambda = 1$  the theorem recovers  $E(n) \rightarrow e - 1$  [5, Theorem 6.8], and in general it proves the exponent conjecture of [11, Problem 8.3].*

*Remark 6.4.* The exponent  $\max(\lambda, 1 - \lambda)$  is minimized at  $\lambda = \frac{1}{2}$ : among fixed-ratio sequences, the balanced ones have the smallest *limiting* ceiling; for the finite-order comparison see Proposition 5.2 and Problem 7.1. Heuristically, the class is controlled near an endpoint out to a distance proportional to that endpoint's share of the data; the exposed region is governed by the larger share. Numerically the maximizer of  $S^{(p,2p)}$  drifts toward the transition  $\lambda = \frac{1}{3}$  from the majority side ( $x^* \approx 0.463, 0.448, 0.440, 0.430, 0.425$  for  $n = 6, 12, 18, 27, 36$ ), while the balanced maximizer sits at  $\frac{1}{2}$  exactly for all orders tested. The scale of the drift is *not* the CLT window: at  $x = \lambda + c/\sqrt{n}$  the two binomial weights of Lemma 6.1(a) stabilize to  $\Phi(\mp c/\sigma)$  with  $\sigma^2 = \lambda(1 - \lambda)$ , giving the transition profile

$$(e^\lambda - 1)\Phi(-c/\sigma) + (e^{1-\lambda} - 1)\Phi(c/\sigma),$$

which for  $\lambda < \frac{1}{2}$  stays strictly below  $e^{1-\lambda} - 1$  at every finite  $c$ ; hence  $\sqrt{n}(x_n^* - \lambda) \rightarrow \infty$ , and balancing the envelope loss in  $e^{1-x}$  against the Gaussian weight deficit suggests the moderate-deviation location  $x_n^* - \lambda \asymp \sqrt{(\log n)/n}$ , with the ceiling deficit on the same scale (Problem 7.2). The balanced case is different: there the transition profile is flat at leading order, the maximizer is (conjecturally) pinned at the midpoint, and the *midpoint* deficit  $\sqrt{e} - 1 - S^{(m,m)}(\frac{1}{2})$  is  $\Theta(n^{-1/2})$ : the upper bound is in [11, Theorem 7.2], and the  $k = 1$  term of the missing-mass sum there, the central-binomial ratio  $\binom{2m-2}{m-1} 2^{1-2m} \asymp m^{-1/2}$ , already gives the matching lower bound. Conjecturally the worst-case deficit shares that scale.

## 7. PROBLEMS

**Problem 7.1.** Maximizer location and finite-order monotonicity. For the balanced split the maximizer of  $S$  is conjecturally the midpoint for every order [11, Problem 8.2]. For asymmetric splits with  $\lambda < \frac{1}{2}$  (say the canonical  $p_n = \lfloor \lambda n \rfloor$ ), prove that the maximizer  $x_n^* \rightarrow \lambda$  from the majority side with  $\sqrt{n}(x_n^* - \lambda) \rightarrow \infty$ , and determine its rescaled asymptotics on the moderate-deviation scale  $\sqrt{(\log n)/n}$  of Remark 6.4. Separately: is  $E^{(p, n-p)}$  nondecreasing in  $|n - 2p|$  at every fixed  $n$ , strictly so beyond the ties of Proposition 5.2? (Verified numerically for  $n \leq 8$ .)

**Problem 7.2.** The ceiling deficit. For canonical sequences  $p_n = \lfloor \lambda n \rfloor$  (the limit  $p_\nu/n_\nu \rightarrow \lambda$  alone cannot determine second-order behavior, since  $E$  is sensitive to  $p_\nu/n_\nu - \lambda$  at the deficit scale), determine the asymptotics of  $e^M - 1 - E^{(p_n, q_n)}$ : for  $\lambda = \frac{1}{2}$  it should be  $\Theta(n^{-1/2})$ , with the constant computable from the midpoint expansion in [11, Theorem 7.2]; for  $\lambda \neq \frac{1}{2}$ , Remark 6.4 suggests  $\asymp \sqrt{(\log n)/n}$ , via the Berry–Esseen approximation of the binomial weights at the transition.

**Problem 7.3.** Explicit chambers at order five. The splits  $(3, 2)$  and  $(2, 3)$  are the first with  $p, q \geq 2$  beyond the balanced order-four case: their charts are quartic nonnegativity conditions with two data anchors. Describe the chambers explicitly, extending [5, §5] and [11, Theorem 5.1].

**Problem 7.4.** Transplant the family. Replace  $D^n$  by a disconjugate operator in Pólya form (quasi-derivative data at both ends, extending [6]), and replace the interval by a grid (extending [7]). In the discrete case, is the forcing set polyhedral for *every* split—strengthening the “curvature is a continuum effect” principle of [7]—and does the ceiling theorem survive with the same exponent?

#### ACKNOWLEDGMENTS

Eighth in a series descending from [4], after [5, 6, 7, 8, 9, 10, 11]; it proves the conjecture of [11, Problem 8.3] and removes the conditionality of [11, Theorem 7.2]. All exact constants were verified in rational arithmetic.

#### REFERENCES

- [1] W. A. Coppel, *Disconjugacy*, Lecture Notes in Mathematics 220, Springer, Berlin, 1971.
- [2] W. Hoeffding, Probability inequalities for sums of bounded random variables, *J. Amer. Statist. Assoc.* **58** (1963), no. 301, 13–30.
- [3] S. Karlin, W. J. Studden, *Tchebycheff Systems: With Applications in Analysis and Statistics*, Pure and Applied Mathematics XV, Interscience, New York, 1966.
- [4] D. Pan, A maximum principle for high-order derivatives, *Amer. Math. Monthly* **120** (2013), no. 9, 846–848.
- [5] D. Pan, Which endpoint data force the maximum principle for high-order derivatives?, preprint, 2026.
- [6] D. Pan, Forcing sets for maximum principles of disconjugate operators, companion note, 2026.
- [7] D. Pan, Which endpoint data force the maximum principle for higher-order convex sequences?, companion note, 2026.
- [8] D. Pan, Which data force the maximum principle for radial polyharmonic functions on a ball?, companion note, 2026.
- [9] D. Pan, Forcing Navier data for polyharmonic maximum principles, companion note, 2026.
- [10] D. Pan, Forcing Dirichlet data for polyharmonic maximum principles, companion note, 2026.
- [11] D. Pan, Which two-point Hermite data force the maximum principle for  $u^{(2m)} \geq 0$ ?, companion note, 2026.
- [12] G. Szegő, *Orthogonal Polynomials*, 4th ed., Amer. Math. Soc. Colloq. Publ. 23, American Mathematical Society, Providence, RI, 1975.