FORCING NAVIER DATA FOR POLYHARMONIC MAXIMUM PRINCIPLES

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Abstract. Fifth in a series descending from [3]: we develop the forcing theory of maximum principles for the class $\Delta^m u \geq 0$ on a bounded, Dirichlet-regular domain $\Omega \subset \mathbb{R}^d$, with prescribed Navier data $u, \Delta u, \ldots, \Delta^{m-1}u$ on $\partial\Omega$ (only the existence of the extremal interpolant uses the boundary regularity; the comparison theorems need boundedness alone). Navier conditions factor the problem into iterated Dirichlet Laplacians, so second-order maximum principles do all the work: the class has a pointwise-greatest element for m odd and a pointwise-least element for m even—the parity phenomenon of [4] resurfacing at the PDE level—and no positivity theory of polyharmonic Green functions is needed anywhere. Consequently, for m odd, Navier data force the maximum principle $u \leq \max_{\partial\Omega} g_0$ precisely when the single polyharmonic Navier interpolant satisfies it; the forcing set is a union of convex cones indexed by boundary points; and the alternating sign condition $(-1)^{m-k} g_k \geq 0$ $(1 \leq k \leq m-1)$ forces unconditionally. For $m = 3$ with constant data the forcing cone is polyhedral with exactly two facets, whose slopes are the *torsion ratios* $\sup_\Omega S/(-T)$ and $\inf_\Omega S/(-T)$, where T is the torsion function ($\Delta T = 1$, $T|_{\partial\Omega} = 0$) and $\Delta S = T$, $S|_{\partial\Omega} = 0$: apparently new domain functionals, equal on the ball $B_R \subset \mathbb{R}^d$ to $(d+4)R^2/\big(4d(d+2)\big)$ and $R^2/\big(d(d+2)\big)$. The worst-case interior overshoot under $|g_k| \leq 1$ is the maximum of the alternating torsion sum $\sum_{k=1}^{m-1} |T_k|$, e.g. $53/384$ for $m = 3$ on the unit interval. In one dimension the Navier interpolant is the Lidstone interpolant and the parity lemma is classical.

1. Introduction

For polyharmonic operators with *Dirichlet* (clamped) boundary conditions, positivity is a famously delicate affair: it holds on balls by Boggio's theorem, and can fail on general domains (see [1]). With *Navier* conditions—prescribing $u, \Delta u, \ldots, \Delta^{m-1}u$ on the boundary—the situation is entirely different: the boundary value problem factors into m second-order Dirichlet problems, and the classical maximum principle can simply be applied m times. This note exploits that factorization to carry the forcing program of [4, 5, 6, 7] to polyharmonic operators on every bounded, Dirichlet-regular domain; the comparison halves of the theory hold on arbitrary bounded domains.

The question, as always in this series: for the differential inequality class, which prescribed data force the maximum principle? Fix $m \geq 1$ and a bounded domain $\Omega \subset \mathbb{R}^d$. For $g = (g_0, \ldots, g_{m-1}) \in C(\partial\Omega)^m$ let

$$\mathcal{K}_m(g) \;=\; \Big\{u \in C^{2m}(\Omega) \;:\; \Delta^k u \in C(\overline{\Omega}) \ (0 \leq k \leq m-1), \quad \Delta^m u \geq 0 \text{ in } \Omega, \quad \Delta^k u|_{\partial\Omega} = g_k\Big\}.$$

Call g *max-forcing* if every $u \in \mathcal{K}_m(g)$ satisfies $u \leq \max_{\partial\Omega} g_0$ on $\overline{\Omega}$, and *min-forcing* if every member satisfies $u \geq \min_{\partial\Omega} g_0$. In [7] we treated radial functions on balls with data at the center; Problem 6.1 there records why center data cannot work beyond the radial setting (an explicit

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polyharmonic family with zero data is unbounded), and points to the present formulation as the repair.

Results: the parity theorem and the reduction of forcing to a single function (Section 2); the alternating criterion and the boundary-indexed cone decomposition (Section 3); the complete constant-data answer for $m = 3$, where two apparently new domain functionals—torsion ratios—appear as the slopes of a two-facet cone (Section 4); the exact worst-case overshoot, an alternating sum of iterated torsion functions (Section 5); and the one-dimensional specialization, where all of this meets Lidstone interpolation (Section 6). Throughout, only the second-order maximum principle is used; the sign pathologies of polyharmonic Green functions never enter.

2. The parity theorem

Two standing hypotheses, used separately. For the comparison statements, Ω is merely a bounded domain. For existence statements we assume

(R) Ω is bounded and every boundary point is regular for the Dirichlet Laplacian,

under which the *Navier interpolant* exists: the function v_g with $\Delta^m v_g = 0$ and Navier data g, built by the cascade $v^{(m-1)} :=$ the Perron solution with boundary data g_{m-1}; then, downward, $v^{(k)}$ the solution of $\Delta v^{(k)} = v^{(k+1)}$, $v^{(k)}|_{\partial\Omega} = g_k$; finally $v_g := v^{(0)}$. Each step solves classically—the right-hand sides are bounded and locally Hölder (indeed locally $C^{2,\alpha}$), so a Newtonian potential plus a Perron correction produces a solution in $C^{2,\alpha}_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$; see [2, §§2.8 and 4.2]. Descending interior regularity upgrades each stage: the top of the cascade is harmonic, hence smooth, and inductively each $v^{(k)}$ solves a Poisson equation with smooth right-hand side, so $v^{(k)} \in C^\infty(\Omega)$; in particular $v_g \in C^{2m}(\Omega)$. Thus $v_g \in \mathcal{K}_m(g)$, and it is the unique function with $\Delta^m v = 0$ and the given data, by m applications of uniqueness for the Dirichlet Laplacian.

Lemma 2.1 (Sign lemma). *Let Ω be a bounded domain and let h be admissible ($h \in C^{2m}(\Omega)$, $\Delta^k h \in C(\overline{\Omega})$ for $k \le m-1$) with $\Delta^m h \ge 0$ in Ω and zero Navier data. Then*

$$(-1)^{m-k}\,\Delta^k h \;\ge\; 0 \quad \text{in } \Omega, \qquad 0 \le k \le m-1;$$

in particular $(-1)^m h \ge 0$.

Proof. Downward induction. $\Delta^{m-1}h$ is subharmonic ($\Delta(\Delta^{m-1}h) = \Delta^m h \ge 0$), continuous on $\overline{\Omega}$, and zero on $\partial\Omega$: the maximum principle gives $\Delta^{m-1}h \le 0$. If $(-1)^{m-k-1}\Delta^{k+1}h \ge 0$, then $\Delta^k h$ is subharmonic or superharmonic according to the sign, vanishes on $\partial\Omega$, and the maximum or minimum principle gives $(-1)^{m-k}\Delta^k h \ge 0$. ∎

Define also the *iterated torsion functions*: under (R), $T_0 \equiv 1$ and, for $j \ge 1$, $\Delta T_j = T_{j-1}$ in Ω, $T_j|_{\partial\Omega} = 0$. By Lemma 2.1 (or directly), $(-1)^j T_j \ge 0$, and by the strong maximum principle $(-1)^j T_j > 0$ in Ω; $T_1 = T$ is the torsion function normalized by $\Delta T = 1$, so $T \le 0$ here.

Theorem 2.2 (Parity, and the forcing reduction). *Let $g \in C(\partial\Omega)^m$ and assume (R).*

(i) *If m is odd, v_g is the pointwise-greatest element of $\mathcal{K}_m(g)$; if m is even, the pointwise-least.*

(ii) *For m odd: g is max-forcing iff $v_g \le \max_{\partial\Omega} g_0$ on Ω; and no g is min-forcing.*

(iii) *For m even: g is min-forcing iff $v_g \ge \min_{\partial\Omega} g_0$ on Ω; and no g is max-forcing.*

(iv) *The max-forcing set (for m odd) is a union of closed convex cones indexed by the boundary:*

$$\mathcal{D}_m(\Omega) \;=\; \bigcup_{x\in\partial\Omega} \mathcal{C}_x, \qquad \mathcal{C}_x = \big\{g:\ g_0 \le g_0(x) \text{ on } \partial\Omega,\ \ v_g \le g_0(x) \text{ on } \Omega\big\},$$

each $\mathcal{C}_x$ cut out by conditions linear in g.

Proof. (i) Apply Lemma 2.1 to $h = u - v_g$. (ii) If $v_g \le \max g_0$, every member is $\le v_g \le \max g_0$; conversely v_g is a member. For the second half, note T_m has zero Navier data ($\Delta^k T_m = T_{m-k}$ vanishes on $\partial\Omega$ for $k \le m-1$) and $\Delta^m T_m = T_0 = 1$, so $u + t\,T_m \in \mathcal{K}_m(g)$ for every $t \ge 0$; since $T_m < 0$ in Ω for m odd, this family is unbounded *below* at interior points: no data can force the minimum principle. (iii) Symmetrically: for m even $T_m > 0$ in Ω, and $u + tT_m$ is unbounded above: no max-forcing; the least element gives the min-forcing criterion. (iv) The benchmark $\max_{\partial\Omega} g_0$ is attained at some $x \in \partial\Omega$; if g is forcing, then $g \in \mathcal{C}_x$ for such x; conversely each $\mathcal{C}_x$ consists of forcing data. Convexity and the cone property: $g \mapsto v_g$ is linear (each cascade step is), and the conditions are pointwise linear inequalities. Closedness: the cascade operators are bounded on sup-norms—$\|H\varphi\|_\infty \le \|\varphi\|_\infty$ by the maximum principle, $\|Gf\|_\infty \le \|f\|_\infty \max_{\overline{\Omega}}(-T)$ by comparison with the torsion function—so v_g depends continuously on g, and each defining inequality survives uniform limits. □

Remark 2.3. The parity is another incarnation of the phenomenon of [4, Remark 6.1], where prescribing k derivatives at the right endpoint produced a greatest element for k odd and a least element for k even; Navier data distributes conditions in the way that alternates the maximum principle m times. Note that (i)–(iii) place no regularity demand on $\partial\Omega$ beyond what (R) needs for v_g and T_m to exist; the comparison halves need only boundedness.

3. The alternating criterion

The descending-data criterion of the earlier papers becomes an alternating sign condition, and needs no solvability at all.

Theorem 3.1 (Alternating criterion). *Let Ω be a bounded domain and $u \in \mathcal{K}_m(g)$. If*

$$(-1)^{m-k}\, g_k \;\ge\; 0 \quad \text{on } \partial\Omega, \qquad 1 \le k \le m-1,$$

then $(-1)^{m-1}\Delta u \ge 0$ in Ω. Consequently such g is max-forcing for every g_0 when m is odd (u is subharmonic), and min-forcing for every g_0 when m is even (u is superharmonic).

Proof. The chain of Lemma 2.1, with the prescribed signs replacing the zeros: $\Delta^{m-1}u$ is subharmonic, so $\Delta^{m-1}u \le \max g_{m-1} \le 0$; inductively, if $(-1)^{m-k-1}\Delta^{k+1}u \ge 0$ then $\Delta^k u$ is sub- or superharmonic accordingly, and the corresponding principle together with $(-1)^{m-k}g_k \ge 0$ propagates the sign, down to $k = 1$. □

4. Order three with constant data: torsion ratios

Let $m = 3$, the first order at which max-forcing is nontrivial: for $m = 1$ every g is max-forcing (subharmonic functions), and for $m = 2$ none is, by Theorem 2.2(iii). Assume (R), and prescribe *constant* Navier data (c_0, c_1, c_2). The cascade gives the interpolant in closed operator-free form:

$$v \;=\; c_0 + c_1\,T + c_2\,S, \qquad \Delta T = 1, \quad \Delta S = T, \quad T|_{\partial\Omega} = S|_{\partial\Omega} = 0,$$

with $T \le 0 \le S$ ($T = T_1$, $S = T_2$). Define the *torsion ratios*

$$\rho_{\max}(\Omega) \;=\; \sup_{\Omega} \frac{S}{-T}, \qquad \rho_{\min}(\Omega) \;=\; \inf_{\Omega} \frac{S}{-T} \;\ge\; 0.$$

Lemma 4.1. *$\rho_{\max}(\Omega) \le \max_{\overline{\Omega}}(-T) < \infty$, and $\rho_{\min}(\Omega) < \rho_{\max}(\Omega)$ for every bounded Ω.*

Proof. For $\lambda \ge \max(-T)$, the function $w = -\lambda T - S$ has $\Delta w = -\lambda - T = -\lambda + (-T) \le 0$ and $w|_{\partial\Omega} = 0$, so $w \ge 0$ by the minimum principle: $S \le \lambda(-T)$, whence $\rho_{\max} \le \lambda$. If ρ were constant, $S = \lambda(-T)$ would give $T = \Delta S = -\lambda\Delta T = -\lambda$, a nonzero constant vanishing on $\partial\Omega$: impossible. □

Theorem 4.2 (The two-facet cone)**.** *Under (R), the constant data* (c_0, c_1, c_2) *is max-forcing if and only if*

$$c_1 \;\geq\; \begin{cases} \rho_{\max}(\Omega)\, c_2, & c_2 \geq 0, \\ \rho_{\min}(\Omega)\, c_2, & c_2 \leq 0. \end{cases}$$

Thus the constant-data forcing cone is polyhedral with exactly two facets, of slopes the torsion ratios, independent of c_0*.*

Proof. By Theorem 2.2(ii), forcing means $v \leq c_0$ on Ω, i.e. $c_1 T + c_2 S \leq 0$; dividing by $-T > 0$ in Ω, this reads $c_1 \geq c_2\, \rho(x)$ for all $x \in \Omega$, where $\rho = S/(-T)$; taking the supremum over x for $c_2 \geq 0$ and the infimum for $c_2 \leq 0$ gives the criterion. The two facets are distinct by Lemma 4.1. □

Example 4.3 (Balls)*.* On $B_R \subset \mathbb{R}^d$: $T = (|x|^2 - R^2)/(2d)$ and $8d^2(d+2)\, S = (R^2 - |x|^2)\big((d+4)R^2 - d\,|x|^2\big)$, so $\rho = \big((d+4)R^2 - d\,|x|^2\big)/\big(4d(d+2)\big)$ is radially decreasing and

$$\rho_{\max}(B_R) = \frac{(d+4)\,R^2}{4d(d+2)} \quad \text{(at the center)}, \qquad \rho_{\min}(B_R) = \frac{R^2}{d(d+2)} \quad \text{(as a boundary limit)}.$$

Example 4.4 (Intervals)*.* On $\Omega = (0,1) \subset \mathbb{R}$: $T = x(x-1)/2$, $S = x(x-1)(x^2 - x - 1)/24$, $\rho = (1 + x - x^2)/12$, so $\rho_{\max} = 5/48$ (at the midpoint) and $\rho_{\min} = 1/12$ (boundary limit)—consistent with Example 4.3 at $d = 1$, $R = \frac{1}{2}$. Concretely: every even-order-data function on $[0,1]$ with $u^{(6)} \geq 0$, $u'' \equiv c_1$, $u^{(4)} \equiv c_2$ on the boundary stays below $\max\{u(0), u(1)\}$ iff $c_1 \geq \frac{5}{48} c_2$ (for $c_2 \geq 0$).

Remark 4.5. $\rho_{\max}$ and $\rho_{\min}$ appear to be new spectral-geometric quantities of Saint-Venant type: they are scale-covariant ($\rho(\lambda\Omega) = \lambda^2 \rho(\Omega)$), monotonicity and extremal properties under symmetrization are unclear, and Problem 7.2 asks for the sharp isoperimetric statements.

5. The overshoot: alternating torsion sums

Write H for harmonic extension ($H\varphi$ = the Perron solution with data φ) and G for the Green operator ($\Delta(Gf) = -f$, $Gf|_{\partial\Omega} = 0$); both are positivity preserving (using $H1 = 1$ and $\Delta(Gf) = -f$), and the cascade reads

$$v_g \;=\; Hg_0 \;+\; \sum_{k=1}^{m-1} (-G)^k H\, g_k, \qquad \text{and} \qquad T_k \;=\; (-G)^k\, 1 \;=\; (-1)^k\, G^k\, 1 \tag{1}$$

Theorem 5.1 (Overshoot)**.** *Let* m *be odd, assume (R), and let* $E_m(\Omega)$ *be the supremum of* $\max_{\overline{\Omega}} u - \max_{\partial\Omega} g_0$ *over all members of all classes with* $|g_k| \leq 1$ *on* $\partial\Omega$ *for* $1 \leq k \leq m-1$ *(*g_0 *unconstrained). Then*

$$E_m(\Omega) \;=\; \max_{\overline{\Omega}} \sum_{k=1}^{m-1} |T_k|,$$

attained by the constant data $g_k = (-1)^k$*,* $g_0 = 0$*, and the interpolant itself. For* $m = 3$*:* $E_3(\Omega) = \max_{\overline{\Omega}}(\,-T + S\,)$*; e.g.* $E_3\big((0,1)\big) = \frac{53}{384}$ *and* $E_3(B_R) = \frac{R^2}{2d} + \frac{(d+4)R^4}{8d^2(d+2)}$*.*

Proof. Upper bound: $u \leq v_g$ (m odd), and in (1) positivity preservation alone gives the bound: $|Hg_k| \leq 1$, hence $|G^k H g_k| \leq G^k |Hg_k| \leq G^k\, 1 = |T_k|$ pointwise, while $Hg_0 \leq \max g_0$. Hence $u - \max g_0 \leq \sum_{k\geq 1} |T_k|$. Attainment: with $g_k = (-1)^k$ and $g_0 = 0$, the interpolant is $v = \sum_{k\geq 1} (-1)^k T_k = \sum_{k \geq 1} |T_k|$, a member of its class with benchmark 0. The two displayed values follow from Examples 4.3 and 4.4 at the maximizing point (the midpoint, resp. the center: both $-T$ and S are radially decreasing there). □

6. One dimension: Lidstone interpolation

For $d = 1$, $\Omega = (a, b)$, Navier data is the set of *even* derivatives $u^{(2k)}(a), u^{(2k)}(b)$, $0 \le k \le m-1$, and the Navier interpolant is the classical *Lidstone interpolant*: the unique polynomial of degree $\le 2m-1$ with those data. Theorem 2.2(i) specializes to: a function with $u^{(2m)} \ge 0$ lies below (for m odd) or above (for m even) its Lidstone interpolant—a sign pattern that is classical in the theory of completely convex functions and Lidstone series, going back to Widder [8, Theorem 1.1]. The forcing theory built on top of it—Theorems 2.2(ii), 4.2 and 5.1—appears to be new even in this one-dimensional case, and complements the one-endpoint data theory of [4]: there the data lived at (a^{n-1}, b); here it is split evenly, and the parity of the split governs which principle can be forced, exactly as [4, Remark 6.1] predicts.

7. Problems

Problem 7.1. Describe the full forcing set $\mathcal{D}_3(\Omega) \subseteq C(\partial\Omega)^3$ for nonconstant data: by Theorem 2.2(iv) it is a boundary-indexed union of convex cones cut by the linear conditions $Hg_0 - GHg_1 + G^2Hg_2 \le g_0(x)$; determine its extreme rays and boundary structure, the infinite-dimensional counterpart of the contact-type stratification of [4, §6].

Problem 7.2. Develop the isoperimetric theory of the torsion ratios: among domains of given volume, which extremize $\rho_{\max}$ and $\rho_{\min}$? Is the ball extremal, as for the Saint-Venant torsional rigidity? Symmetrization acts on T and S in the same direction, so the ratio is not obviously monotone; even upper and lower bounds in terms of inradius and volume would be of interest.

Problem 7.3. Dirichlet (clamped) data: on domains where Boggio-type positivity holds—balls, and their perturbations [1]—does the class $\Delta^m u \ge 0$ with prescribed $u, \partial_\nu u, \dots, \partial_\nu^{m-1} u$ on $\partial\Omega$ have a one-signed comparison with its polyharmonic interpolant, and a forcing theory? Here the Green-function sign is genuinely needed, in contrast to everything above; the failure of positivity on general domains should manifest as the failure of the greatest-element property.

Problem 7.4. Interpolate between this paper and [7]: on the ball, mixed data prescribing some $\Delta^k u$ at the center and the rest on the boundary. Which mixtures admit one-signed comparison? The counterexample of [7, Problem 6.1] shows that too much center data fails; Navier data is the all-boundary extreme.

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Fifth in a series descending from [3], after [4, 5, 6, 7]. The formulation of this paper was forced, in both senses, by a counterexample of the review process recorded in [7, Problem 6.1].

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