A Maximum Principle for High-Order Derivatives

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Abstract. We prove a maximum principle for high-order derivatives under initial conditions.

1 Introduction

We begin with a well-known definition.

Definition 1.1. The *maximum principle* holds for $f(x)$ on $[a, b]$ if $f(x) \leq \max\{f(a), f(b)\}$ on $[a, b]$.

It is well known that $f'(x) \geq 0$ implies the maximum principle because of nondecreasingness, and that $f''(x) \geq 0$ implies the maximum principle because of nonconcavity. It is also well known that $f^{(n)}(x) \geq 0$ where $n \geq 3$ does not necessarily imply the maximum principle. For example, consider $f(x) = \pm x^2$ and its third derivative.

We present conditions under which $f^{(n)}(x) \geq 0$ where $n \geq 3$ does imply the maximum principle. Let $I = [a, b]$ be a closed interval of the real line, and let $C^n(I)$ be the set of real-valued functions on I that are n-times continuously differentiable. Recall that

$$P_n(x) := \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x - a)^i$$

is the nth-degree Taylor polynomial of $f(x)$ at a, where $f(x) \in C^n(I)$ and $a \in I$.

Theorem 1.1. *Let $f(x) \in C^n([a, b])$ for some $n \geq 2$. If $f^{(n)}(x) \geq 0$ on $[a, b]$, then $f(x) \leq v_n(x)$ on $[a, b]$, where*

$$v_n(x) := P_{n-2}(x) + \frac{f(b) - P_{n-2}(b)}{(b - a)^{n-1}}(x - a)^{n-1}.$$

The following theorem is a corollary of Theorem 1.1. It shows that the maximum principle holds under initial conditions.

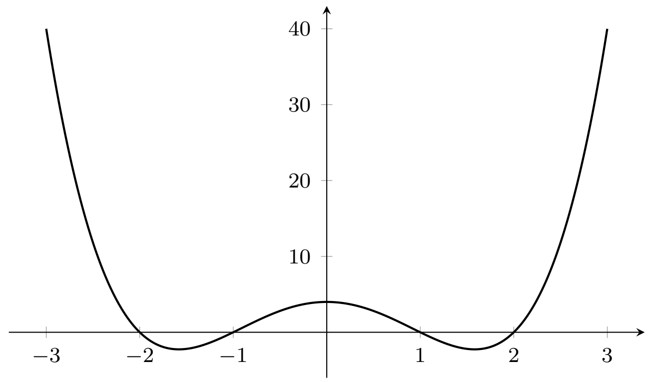
Theorem 1.2. *Let $f(x) \in C^n([a, b])$ for some $n \geq 3$. If*

(i) $f^{(n)}(x) \geq 0$ on $[a, b]$, and

(ii) $f(b) \geq P_i(b)$ for $i = 1, \ldots, n - 2$,

then $f(x) \leq \max\{f(a), f(b)\}$ on $[a, b]$ (i.e., the maximum principle holds for $f(x)$ on $[a, b]$).

Example 1.1. Below is a graph of $f(x) = (x - 2)(x - 1)(x + 1)(x + 2) = x^4 - 5x^2 + 4$.



It is false that $f'(x), f''(x) \geq 0$ on $[0,3]$, but it is true that $f'''(x) \geq 0$ on $[0,3]$. Also, $f(3) = 40 \geq P_1(3) = 4$. Therefore, by Theorem 1.2, the maximum principle holds for $f(x)$ on $[0,3]$.

Proof of Theorem 1.1. Suppose that $f(x) \in C^n([a,b])$ for some $n \geq 2$ and $f^{(n)}(x) \geq 0$ on $[a,b]$. By Taylor's theorem,

$$g(x) := \frac{f(x) - P_{n-2}(x)}{(x-a)^{n-1}} = \frac{1}{(n-2)!} \int_0^1 (1-t)^{n-2} f^{(n-1)}(a + t(x-a))\, dt.$$

It follows that

$$g'(x) = \frac{1}{(n-2)!} \int_0^1 t(1-t)^{n-2} f^{(n)}(a + t(x-a))\, dt.$$

Since $f^{(n)}(x) \geq 0$ on $[a,b]$, $g'(x) \geq 0$ on $[a,b]$, so $g(x) \leq g(b)$ on $[a,b]$, which implies

$$f(x) \leq P_{n-2}(x) + \frac{f(b) - P_{n-2}(b)}{(b-a)^{n-1}}(x-a)^{n-1} = v_n(x)$$

on $[a,b]$. □

Proof of Theorem 1.2. We prove Theorem 1.2 by mathematical induction.

First, we prove the initial case $n = 3$. Suppose that for some $f(x) \in C^3([a,b])$, we have $f'''(x) \geq 0$ on $[a,b]$ and $f(b) \geq P_1(b)$. By Theorem 1.1, we have

$$f(x) \leq v_3(x) = P_1(x) + \frac{f(b) - P_1(b)}{(b-a)^2}(x-a)^2$$

on $[a,b]$. Because $P_1''(x) = 0$ and $f(b) - P_1(b) \geq 0$, we have $v_3''(x) \geq 0$, which, by convexity, implies that $v_3(x) \leq \max\{v_3(a), v_3(b)\}$ on $[a,b]$. It is obvious that $v_3(a) = f(a)$ and $v_3(b) = f(b)$, so $f(x) \leq v_3(x) \leq \max\{f(a), f(b)\}$ on $[a,b]$. Therefore, the theorem holds in the initial case $n = 3$.

Now, we assume that the theorem holds in the case $n = k$ where $k \geq 3$, and we want to show that the theorem holds in the case $n = k+1$. To do this, suppose that for some $f(x) \in C^{k+1}([a,b])$, we have $f^{(k+1)}(x) \geq 0$ on $[a,b]$ and $f(b) \geq P_i(b)$ for $i = 1, \dots, k-1$. By Theorem 1.1, we have

$$f(x) \leq v_{k+1}(x) = P_{k-1}(x) + \frac{f(b) - P_{k-1}(b)}{(b-a)^k}(x-a)^k$$

on $[a,b]$. Because $P_{k-1}^{(k)}(x) = 0$ and $f(b) - P_{k-1}(b) \geq 0$, we have $v_{k+1}^{(k)}(x) \geq 0$. It is obvious that $v_{k+1}^{(i)}(a) = f^{(i)}(a)$ for $i = 0, \dots, k-1$ and $v_{k+1}(b) = f(b)$, so

$$v_{k+1}(b) = f(b) \geq P_j(b) = \sum_{i=0}^{j} \frac{f^{(i)}(a)}{i!}(b-a)^i = \sum_{i=0}^{j} \frac{v_{k+1}^{(i)}(a)}{i!}(b-a)^i$$

for $j = 1, \ldots, k-2$. By the induction assumption,

$$v_{k+1}(x) \leq \max\{v_{k+1}(a), v_{k+1}(b)\} = \max\{f(a), f(b)\}$$

on $[a, b]$. Therefore, $f(x) \leq v_{k+1}(x) \leq \max\{f(a), f(b)\}$ on $[a, b]$. This concludes the mathematical induction, and the proof is complete. $\square$