A Maximum Principle for High Order Derivatives

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1 Introduction

We begin with some well-known definitions.

Definition 1.1. The maximum principle holds for $f(x)$ on $[a, b]$ if $f(x) \leq \max\{f(a), f(b)\}$ on $[a, b]$.

It is well known that $f'(x) \geq 0$ implies the maximum principle because of nondecreasingness, and that $f''(x) \geq 0$ implies the maximum principle because of nonconcavity. It is also well known that $f^{(n)}(x) \geq 0$ where $n \geq 3$ does not necessarily imply the maximum principle. For example, consider $f(x) = \pm x^2$ and its third derivative.

We present conditions for when $f^{(n)}(x) \geq 0$ where $n \geq 3$ does imply the maximum principle.

Definition 1.2. Recall that

$$P_n(x) := \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

is the nth-degree Taylor polynomial of $f(x)$ at a, where $f(x) \in C^n$ on some interval I and $a \in I$.

Theorem 1.1. *Let $f(x) \in C^n([a, b])$ for some $n \geq 2$. If $f^{(n)}(x) \geq 0$ on $[a, b]$, then $f(x) \leq v_n(x)$ on $[a, b]$, where*

$$v_n(x) := P_{n-2}(x) + \frac{f(b) - P_{n-2}(b)}{(b-a)^{n-1}}(x-a)^{n-1}.$$

The following theorem is a corollary of Theorem 1.1. It shows that the maximum principle holds under initial conditions.

Theorem 1.2. *Let $f(x) \in C^n([a, b])$ for some $n \geq 3$. If*

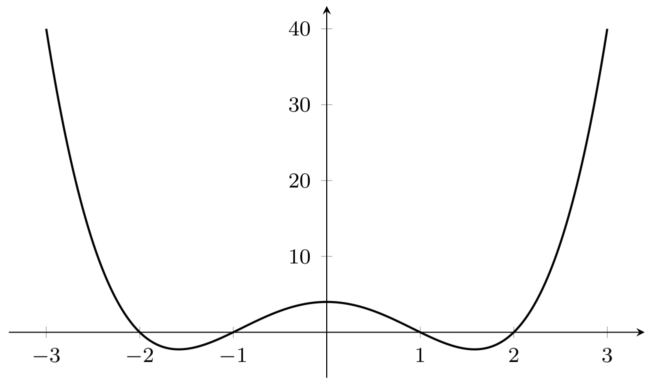
(i) $f^{(n)}(x) \geq 0$ on $[a, b]$ and

(ii) $f(b) \geq P_i(b)$ for $i = 1, \ldots, n-2$,

then $f(x) \leq \max\{f(a), f(b)\}$ on $[a, b]$ (i.e. the maximum principle holds for $f(x)$ on $[a, b]$).

The following is an example of a function that satisfies a maximum principle by Theorem 1.2.

Example 1.1. Below is a graph of $f(x) = (x-2)(x-1)(x+1)(x+2) = x^4 - 5x^2 + 4$.



It is false that $f'(x), f''(x) \geq 0$ on $[0,3]$, but it is true that $f'''(x) \geq 0$ on $[0,3]$. Also, $f(3) = 40 \geq P_1(3) = 4$. Therefore, by Theorem 1.2, the maximum principle holds for $f(x)$ on $[0,3]$.

2 Proof of Theorem 1.1

First we prove some lemmas needed for the proof of Theorem 1.1.

Lemma 2.1. *If $f(x) \in C^{n+1}$ on some interval I and $x, a \in I$, then*

$$f(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i + \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,dt = P_n(x) + R_n(x).$$

We omit the proof for Lemma 2.1 because it is well known; one proof involves the fundamental theorem of calculus, integration by parts, and mathematical induction. ($R_n(x)$ is called the remainder of the Taylor series.)

Lemma 2.2. *$G_n(x,t) \leq 0$ for all $n \geq 1$, where*

$$G_n(x,t) := \begin{cases} (x-t)^n - \dfrac{(x-a)^n(b-t)^n}{(b-a)^n} & \text{if } a \leq t < x \leq b \\ -\dfrac{(x-a)^n(b-t)^n}{(b-a)^n} & \text{if } a \leq x \leq t \leq b \end{cases}$$

is a piecewise function defined on the square $[a,b] \times [a,b]$.

Proof. Let n be a positive integer.

Case 1. Suppose $a \leq t < x \leq b$.

$$\begin{aligned} (x-t)^n - \frac{(x-a)^n(b-t)^n}{(b-a)^n} &= \frac{(x-t)^n(b-a)^n - (x-a)^n(b-t)^n}{(b-a)^n} \\ &= \frac{(x-t)(b-a) - (x-a)(b-t)}{(b-a)^n} \\ &\qquad \sum_{i=0}^{n-1}(x-t)^{n-1-i}(b-a)^{n-1-i}(x-a)^i(b-t)^i \\ &= \frac{(x-b)(t-a)}{(b-a)^n} \end{aligned}$$

$$\sum_{i=0}^{n-1}(x-t)^{n-1-i}(b-a)^{n-1-i}(x-a)^i(b-t)^i$$
$$\leq 0$$

because $x-b \leq 0$ and $t-a, x-t, b-a, x-a, b-t \geq 0$. So $G_n(x,t) \leq 0$ for $a \leq t < x \leq b$ and $n \geq 1$.

Case 2. Suppose $a \leq x \leq t \leq b$. Then $x-a, b-t, b-a \geq 0$, which implies $-\frac{(x-a)^n(b-t)^n}{(b-a)^n} \leq 0$. So $G_n(x,t) \leq 0$ for $a \leq x \leq t \leq b$ and $n \geq 1$.

We have proved all the cases, so the lemma holds. □

Now we prove Theorem 1.1.

Proof. We prove Theorem 1.1 by mathematical induction.

First we prove the initial case $n = 2$. Suppose $f(x) \in C^2([a,b])$ and $f''(x) \geq 0$ on $[a,b]$. By Lemma 2.1, we have

$$f(x) = f(a) + f'(a)(x-a) + \int_a^x (x-t)f''(t)\,dt \tag{1}$$

Let $x = b$ in (1) and solve for $f'(a)$.

$$f(b) = f(a) + f'(a)(b-a) + \int_a^b (b-t)f''(t)\,dt$$
$$f'(a) = \frac{1}{b-a}\left(f(b) - f(a) - \int_a^b (b-t)f''(t)\,dt\right) \tag{2}$$

Substitute (2) for $f'(a)$ in (1).

$$\begin{aligned}
f(x) &= f(a) + \frac{x-a}{b-a}\left(f(b) - f(a) - \int_a^b (b-t)f''(t)\,dt\right) + \int_a^x (x-t)f''(t)\,dt \\
&= f(a) + \frac{f(b)-f(a)}{b-a}(x-a) - \int_a^b \frac{(x-a)(b-t)}{b-a}f''(t)\,dt \\
&\quad + \int_a^x (x-t)f''(t)\,dt \\
&= v_2(x) + \int_a^x \left(x - t - \frac{(x-a)(b-t)}{b-a}\right) f''(t)\,dt \\
&\quad - \int_x^b \frac{(x-a)(b-t)}{b-a}f''(t)\,dt \\
&= v_2(x) + \int_a^b G_1(x,t)f''(t)\,dt.
\end{aligned}$$

Since $G_1(x,t) \leq 0$ by Lemma 2.2 and $f''(x) \geq 0$ on $[a,b]$, $\int_a^b G_1(x,t)f''(t)\,dt \leq 0$, which implies $f(x) \leq v_2(x)$ on $[a,b]$. Therefore the theorem holds in the case $n = 2$.

Now we assume that the theorem holds in the case $n = k$ where $k \geq 2$; that is, we assume that for some $f(x) \in C^k([a,b])$ where $k \geq 2$, $f(x) \leq v_k(x)$ on $[a,b]$ if $f^{(k)}(x) \geq 0$ on $[a,b]$. We want to show from this assumption that the theorem holds in the case $n = k+1$.

Suppose that for some $f(x) \in C^{k+1}([a,b])$, $f^{(k+1)}(x) \geq 0$ on $[a,b]$. By Lemma 2.1, we have

$$f(x) = P_k(x) + \int_a^x \frac{(x-t)^k}{k!} f^{(k+1)}(t)\,dt$$

$$= P_{k-1}(x) + \frac{f^{(k)}(a)}{k!}(x-a)^k + \int_a^x \frac{(x-t)^k}{k!} f^{(k+1)}(t)\, dt. \tag{3}$$

Let $x = b$ in (3) and solve for $f^{(k)}(a)$.

$$f(b) = P_{k-1}(b) + \frac{f^{(k)}(a)}{k!}(b-a)^k + \int_a^b \frac{(b-t)^k}{k!} f^{(k+1)}(t)\, dt.$$

$$f^{(k)}(a) = \frac{k!}{(b-a)^k}\left(f(b) - P_{k-1}(b) - \int_a^b \frac{(b-t)^k}{k!} f^{(k+1)}(t)\, dt\right). \tag{4}$$

Substitute (4) for $f^{(k)}(a)$ in (3).

$$\begin{aligned}
f(x) &= P_{k-1}(x) + \frac{(x-a)^k}{(b-a)^k}\left(f(b) - P_{k-1}(b) - \int_a^b \frac{(b-t)^k}{k!} f^{(k+1)}(t)\, dt\right) \\
&\quad + \int_a^x \frac{(x-t)^k}{k!} f^{(k+1)}(t)\, dt \\
&= P_{k-1}(x) + \frac{f(b) - P_{k-1}(b)}{(b-a)^k}(x-a)^k \\
&\quad + \frac{1}{k!}\left(-\int_a^b \frac{(x-a)^k(b-t)^k}{(b-a)^k} f^{(k+1)}(t)\, dt + \int_a^x (x-t)^k f^{(k+1)}(t)\, dt\right) \\
&= v_{k+1}(x) + \frac{1}{k!}\left(\int_a^x \left((x-t)^k - \frac{(x-a)^k(b-t)^k}{(b-a)^k}\right) f^{(k+1)}(t)\, dt\right. \\
&\quad \left. - \int_x^b \frac{(x-a)^k(b-t)^k}{(b-a)^k} f^{(k+1)}(t)\, dt\right) \\
&= v_{k+1}(x) + \frac{1}{k!}\int_a^b G_k(x,t) f^{(k+1)}(t)\, dt.
\end{aligned}$$

Since $G_k(x,t) \leq 0$ by Lemma 2.2 and $f^{(k+1)}(x) \geq 0$ on $[a,b]$, $\frac{1}{k!}\int_a^b G_k(x,t) f^{(k+1)}(t)\, dt \leq 0$, which implies $f(x) \leq v_{k+1}(x)$ on $[a,b]$. This concludes the mathematical induction. □

3 Proof of Theorem 1.2

Proof. We prove Theorem 1.2 by mathematical induction.

First we prove the initial case $n = 3$. Suppose that for some $f(x) \in C^3([a,b])$, $f'''(x) \geq 0$ on $[a,b]$ and $f(b) \geq P_1(b)$. By Theorem 1.1, we have

$$f(x) \leq v_3(x) = P_1(x) + \frac{f(b) - P_1(b)}{(b-a)^2}(x-a)^2$$

on $[a,b]$.

$v_3''(x) = \frac{f(b)-P_1(b)}{(b-a)^2} \geq 0$ because $f(b) - P_1(b), b - a \geq 0$, so $v_3(x) \leq \max\{v_3(a), v_3(b)\}$ on $[a,b]$. It is obvious that $v_3(a) = f(a)$ and $v_3(b) = f(b)$, so $f(x) \leq v_3(x) \leq \max\{f(a), f(b)\}$ on $[a,b]$. Therefore the theorem holds in the initial case $n = 3$.

Now we assume that the theorem holds in the case $n = k$ where $k \geq 3$; that is, we assume that the maximum principle holds on $[a,b]$ for some $f(x) \in C^k([a,b])$ where $k \geq 2$ if $f^{(k)}(x) \geq 0$ on $[a,b]$ and $f(b) \geq P_i(b)$ for $i = 1, \ldots, k-2$. We want to show from this assumption that the theorem holds in the case $n = k+1$.

Suppose that for some $f(x) \in C^{k+1}([a,b])$, $f^{(k+1)}(x) \geq 0$ on $[a,b]$ and $f(b) \geq P_i(b)$ for $i = 1, \ldots, k-1$. By Theorem 1.1, we have

$$f(x) \leq v_{k+1}(x) = P_{k-1}(x) + \frac{f(b) - P_{k-1}(b)}{(b-a)^k}(x-a)^k$$

on $[a,b]$.

We want to show that $v_{k+1}(x)$ satisfies the kth maximum principle on $[a,b]$. Since $f(b) - P_{k-1}(b) \geq 0$, $v_{k+1}^{(k)}(x) = \frac{f(b)-P_{k-1}(b)}{(b-a)^k} \geq 0$. It is obvious that $v_{k+1}^{(i)}(a) = f^{(i)}(a)$ for $i = 0, \ldots, k-1$ and $v_{k+1}(b) = f(b)$, so

$$v_{k+1}(b) = f(b) \geq P_j(b) = \sum_{i=0}^{j} \frac{f^{(i)}(a)}{i!}(b-a)^i = \sum_{i=0}^{j} \frac{v_{k+1}^{(i)}(a)}{i!}(b-a)^i$$

for $j = 1, \ldots, k-2$, which implies $v_{k+1}(x) \leq \max\{v_{k+1}(a), v_{k+1}(b)\} = \max\{f(a), f(b)\}$ on $[a,b]$.

Therefore, $f(x) \leq v_{k+1}(x) \leq \max\{f(a), f(b)\}$ on $[a,b]$. This concludes the mathematical induction. $\square$