WHICH ENDPOINT DATA FORCE THE MAXIMUM PRINCIPLE FOR HIGH-ORDER DERIVATIVES?

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ABSTRACT. An earlier note of the author [*Amer. Math. Monthly* **120** (2013) 846–848] showed that a function with $f^{(n)} \geq 0$ on $[a, b]$ satisfies the maximum principle $f \leq \max\{f(a), f(b)\}$ provided $f(b)$ dominates the values at b of the Taylor polynomials of f at a of orders $1, \dots, n-2$. Those conditions are sufficient but not necessary. Here we determine exactly which endpoint data $\big(f(a), f'(a), \dots, f^{(n-2)}(a);\, f(b)\big)$ force the maximum principle. The class of admissible functions has a pointwise-greatest element—an explicit polynomial of degree at most $n-1$—so the forcing set $\mathcal{D}_n$ is linearly isomorphic to the set $\mathcal{M}_{n-1}$ of polynomials of degree at most $n-1$ whose maximum on $[0, 1]$ is attained at an endpoint. This yields: (i) $\mathcal{D}_n$ is the union of two closed convex cones ("maximum on the left" and "maximum on the right"), each semidefinite-representable; (ii) for $n = 3$, the sharp single-inequality characterization $f'(a)(b-a) \leq 2\max\{f(b) - f(a), 0\}$; (iii) the criterion $f'(a) \leq 0, \dots, f^{(n-2)}(a) \leq 0$, which forces the maximum principle for *every* value of $f(b)$; (iv) a geometric dichotomy: $\mathcal{D}_3$ is a union of two half-spaces, while for every $n \geq 4$ the cone $\mathcal{D}_n$ is not a finite union of convex polyhedra; nevertheless we determine $\mathcal{D}_4$ completely—its non-forcing complement consists of two mirror-image curved lenses, one for each sign of the leading coefficient of the extremal cubic—and confine the boundary of the forcing cone, for every n, to four explicit algebraic families of at most linearly growing degree, each bounding along a single connected wall, with the closure order of all boundary strata determined combinatorially; and (v) an exact overshoot theorem: the largest possible excess of f over $\max\{f(a), f(b)\}$, given $|f^{(i)}(a)|\,(b-a)^i \leq 1$ for $1 \leq i \leq n-2$, equals the maximum of an explicit polynomial P_n generalizing $x - x^2$; it is $1/4$ for $n = 3$ and $(28 + 19\sqrt{19})/243$ for $n = 4$, and it increases strictly to $e - 1$.

1. INTRODUCTION

Say that $f\colon [a, b] \to \mathbb{R}$ satisfies the *maximum principle* on $[a, b]$ if

$$f(x) \;\leq\; \max\{f(a), f(b)\} \qquad \text{for all } x \in [a, b].$$

Every function with $f' \geq 0$ satisfies it (monotonicity), as does every function with $f'' \geq 0$ (convexity), but for $n \geq 3$ the condition $f^{(n)} \geq 0$ alone does not suffice: $f(x) = x - x^2$ on $[0, 1]$ has $f''' \equiv 0$ and an interior maximum. The note [7] gave endpoint conditions restoring the principle.

Theorem 1.1 ([7, Theorem 1.2])**.** *Let $n \geq 3$ and $f \in C^n([a, b])$ with $f^{(n)} \geq 0$ on $[a, b]$. If*

$$f(b) \;\geq\; P_i(b) \qquad \text{for } i = 1, \dots, n-2,$$

where P_i is the ith Taylor polynomial of f at a, then f satisfies the maximum principle on $[a, b]$.

These conditions are not necessary. The function $f(x) = -x^2$ on $[0, 1]$ has $f''' \equiv 0$ and $f(1) = -1 < 0 = P_1(1)$, yet $f \leq 0 = \max\{f(0), f(1)\}$. One might object that a single function proves little: perhaps this f is merely lucky, while some other function with the same data $\big(f(0), f'(0); f(1)\big) = (0, 0; -1)$ misbehaves. It does not: we will see (Theorem 3.1) that *every f*

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with $f''' \geq 0$ and this data satisfies the maximum principle. The conditions of Theorem 1.1 thus miss part of the true forcing region, and the right question is:

> *For which data* $\big(f(a), f'(a), \ldots, f^{(n-2)}(a);\ f(b)\big)$ *does* $f^{(n)} \geq 0$ *force the maximum principle?*

Both the hypothesis $f^{(n)} \geq 0$ and the conclusion are invariant under the affine substitution $x = a + (b-a)t$, the data transforming by $f^{(i)}(a) \mapsto f^{(i)}(a)(b-a)^i$; so we normalize $[a,b] = [0,1]$ once and for all. It is convenient—and costs nothing, by Remark 2.3—to work in a class one derivative wider than C^n.

Definition 1.2. Fix $n \geq 2$. For a *data vector* $d = (c_0, c_1, \ldots, c_{n-2};\ \beta) \in \mathbb{R}^n$ put

$$\mathcal{K}_n(d) \ = \ \Big\{ f \in C^{n-1}([0,1]) :\ f^{(n-1)} \text{ nondecreasing}, \quad f^{(i)}(0) = c_i \ (0 \leq i \leq n-2), \quad f(1) = \beta \Big\}.$$

The data d is *forcing* if every $f \in \mathcal{K}_n(d)$ satisfies the maximum principle on $[0,1]$. We write $\mathcal{D}_n \subseteq \mathbb{R}^n$ for the set of forcing data.

The class $\mathcal{K}_n(d)$ contains every $f \in C^n$ with $f^{(n)} \geq 0$ and the given data. For $n = 2$ every data vector is forcing (a convex function lies below its chord), so the story begins at $n = 3$. Our results:

- **Reduction** (Theorem 2.2, Corollary 2.5). $\mathcal{K}_n(d)$ has a pointwise-greatest element, the explicit polynomial v_d of degree $\leq n-1$; consequently d is forcing if and only if v_d itself satisfies the maximum principle, and $\mathcal{D}_n$ is linearly isomorphic to the set $\mathcal{M}_{n-1}$ of polynomials of degree $\leq n-1$ with endpoint maximum on $[0,1]$.
- **Two regimes** (Theorem 2.6). $\mathcal{D}_n$ is the union of two closed convex cones—data forcing the maximum to stay at the left endpoint, and at the right endpoint—each cut out by linear inequalities and semidefinite-representable.
- **The case** $n = 3$ (Theorem 3.1). $d = (c_0, c_1; \beta)$ is forcing if and only if
$$c_1 \ \leq \ 2 \max\{\beta - c_0,\ 0\};$$
un-normalized: $f'(a)(b-a) \leq 2\max\{f(b) - f(a), 0\}$. The constant 2 doubles the threshold implicit in Theorem 1.1, and the positive part makes nonpositive initial slopes unconditionally forcing.
- **Descending data** (Theorem 4.1). If $f^{(i)}(a) \leq 0$ for $1 \leq i \leq n-2$, the maximum principle holds *for every value of* $f(b)$—a criterion disjoint from Theorem 1.1.
- **Geometry** (Proposition 5.1, Theorem 5.2). $\mathcal{D}_3$ is a union of two closed half-spaces; but for every $n \geq 4$ the closed cone $\mathcal{D}_n$ is *not* a finite union of convex polyhedra: we compute a two-dimensional affine slice of $\mathcal{D}_n$ exactly, and its boundary contains a parabolic arc. For $n = 4$ we then assemble the complete description of $\mathcal{D}_4$ (Theorem 5.6): the slice, its mirror image under $x \mapsto 1 - x$, and the quadratic locus that glues them. For every degree we characterize the boundary of $\mathcal{M}_k$—four algebraic families, of degree at most $2k-2$ (Theorem 6.2)—and stratify it by contact types: every stratum is connected, exactly four have codimension one, and the stratum count grows from linear (generic contacts) to exponential (all contact patterns) (Theorem 6.3); the closure order of the strata is the evident combinatorial degeneration order (Theorem 6.4).
- **Exact overshoot** (Theorem 6.8). The largest excess of f over $\max\{f(a), f(b)\}$, given $|f^{(i)}(a)|\,(b-a)^i \leq 1$ for $1 \leq i \leq n-2$, is exactly $\max_{[0,1]} P_n$ for an explicit polynomial family P_n generalizing $x - x^2$: it equals $\frac{1}{4}$ for $n = 3$ and $\frac{28+19\sqrt{19}}{243}$ for $n = 4$, and increases strictly to $e - 1$.

A word on context. Functions with $f^{(n)} \geq 0$ are the smooth representatives of the *n-convex* functions of Popoviciu [9] (see Bullen [3]; for a recent survey of the case $n = 3$ and its open problems, see Marinescu–Niculescu [6]), and the inequality $f \le v_d$ underlying our reduction is equivalent to the error formula for Hermite interpolation at the nodes 0 (multiplicity $n-1$) and 1 (multiplicity 1), whose node polynomial $x^{n-1}(x-1)$ has constant sign on $[0,1]$ (see [4, Chapter 3]). The proof below, following [7], is elementary and self-contained, and gives the inequality in the wider class of Definition 1.2. What is new here is the use of the interpolant as a *barrier*: the greatest element of the class turns an infinite-dimensional forcing problem into a finite, and in several cases completely explicit, question about a single polynomial. None of the core arguments below actually require polynomials: the greatest-element reduction, the descending-data criterion, the two-cone decomposition, and the boundary characterization extend verbatim to disconjugate operators in Pólya form—where the sharp constant 2 of Theorem 3.1 becomes the reciprocal of a generalized trapezoidal weight—for which see the companion note [8].

2. The extremal polynomial and the two convex cones

For $d = (c_0, \dots, c_{n-2}; \beta)$ write

$$T_d(x) \;=\; \sum_{i=0}^{n-2} \frac{c_i}{i!}\, x^i, \qquad v_d(x) \;=\; T_d(x) + \big(\beta - T_d(1)\big)\, x^{n-1}.$$

Lemma 2.1. *Let $n \ge 2$ and let $f \in C^{n-1}([0,1])$ with $f^{(n-1)}$ nondecreasing. Then $f \le v_d$ on $[0,1]$, where d is the data vector of f.*

Proof. Taylor's theorem with integral remainder, valid for $f \in C^{n-1}$, gives for $x \in [0,1]$

$$f(x) \;=\; T_d(x) + \frac{1}{(n-2)!} \int_0^x (x-s)^{n-2}\, f^{(n-1)}(s)\, ds,$$

and the substitution $s = tx$ turns the remainder into $x^{n-1} g(x)$ with

$$g(x) \;=\; \frac{1}{(n-2)!} \int_0^1 (1-t)^{n-2}\, f^{(n-1)}(tx)\, dt.$$

If $0 \le x_1 \le x_2 \le 1$ then $f^{(n-1)}(tx_1) \le f^{(n-1)}(tx_2)$ for every $t \in [0,1]$, so g is nondecreasing. Hence for $x \in (0,1]$,

$$f(x) \;=\; T_d(x) + x^{n-1} g(x) \;\le\; T_d(x) + x^{n-1} g(1) \;=\; T_d(x) + \big(\beta - T_d(1)\big) x^{n-1} \;=\; v_d(x),$$

using $g(1) = f(1) - T_d(1) = \beta - T_d(1)$; at $x = 0$ both sides equal c_0. □

This is the argument of [7, Theorem 1.1] with the differentiation under the integral sign replaced by monotonicity of the integrand, which lowers the regularity requirement by one derivative.

Theorem 2.2 (Extremal reduction)**.** *Let $n \ge 2$ and $d \in \mathbb{R}^n$.*

(i) *$v_d \in \mathcal{K}_n(d)$, and $f \le v_d$ on $[0,1]$ for every $f \in \mathcal{K}_n(d)$. Thus v_d is the pointwise-greatest element of $\mathcal{K}_n(d)$; in particular $\mathcal{K}_n(d) \neq \emptyset$.*

(ii) *$d \in \mathcal{D}_n$ if and only if $v_d(x) \le \max\{c_0, \beta\}$ for all $x \in [0,1]$; equivalently, if and only if the polynomial v_d itself satisfies the maximum principle on $[0,1]$.*

Proof. (i) v_d is a polynomial of degree $\le n-1$, so $v_d^{(n-1)}$ is constant, hence nondecreasing. Since x^{n-1} vanishes to order $n-1$ at 0, we get $v_d^{(i)}(0) = T_d^{(i)}(0) = c_i$ for $0 \le i \le n-2$, and $v_d(1) = T_d(1) + \beta - T_d(1) = \beta$. So $v_d \in \mathcal{K}_n(d)$, and Lemma 2.1 gives $f \le v_d$ for every member f.

(ii) If $v_d \le \max\{c_0, \beta\}$ on $[0,1]$, then every $f \in \mathcal{K}_n(d)$ satisfies $f \le v_d \le \max\{c_0, \beta\} = \max\{f(0), f(1)\}$, so d is forcing. Conversely, if d is forcing, then the member $v_d \in \mathcal{K}_n(d)$ must itself satisfy the maximum principle, i.e. $v_d \le \max\{v_d(0), v_d(1)\} = \max\{c_0, \beta\}$. □

Remark 2.3. Since the extremal element v_d is a polynomial, it lies in every reasonable subclass; in particular, replacing $\mathcal{K}_n(d)$ in Definition 1.2 by the smaller class $\{f \in C^n : f^{(n)} \geq 0,\ \text{data } d\}$ produces the same forcing set $\mathcal{D}_n$. All results below are insensitive to this choice.

Remark 2.4. The reduction is quantitative, not merely qualitative: by Theorem 2.2(i), for every x the supremum of $f(x)$ over the class is exactly $v_d(x)$, so the worst-case excess is

$$\sup_{f \in \mathcal{K}_n(d)} \ \max_{[0,1]} f \ - \ \max\{c_0, \beta\} \ = \ \max_{[0,1]} v_d \ - \ \max\{c_0, \beta\},$$

and it is attained.

Let $\mathbb{R}_{\leq k}[x]$ denote the polynomials of degree at most k and

$$\mathcal{M}_k \ = \ \Big\{ p \in \mathbb{R}_{\leq k}[x] \ : \ \max_{[0,1]} p = \max\{p(0), p(1)\} \Big\}.$$

Corollary 2.5 (Polynomial model)**.** *The linear map* data: $\mathbb{R}_{\leq n-1}[x] \to \mathbb{R}^n$,

$$p \ \longmapsto \ \big(p(0), p'(0), \ldots, p^{(n-2)}(0);\, p(1)\big),$$

is an isomorphism with inverse $d \mapsto v_d$, and

$$\mathcal{D}_n \ = \ \mathrm{data}\,(\mathcal{M}_{n-1}).$$

Proof. Both spaces have dimension n. If $\mathrm{data}(p) = 0$ then $p = c\,x^{n-1}$ for some c (all derivatives at 0 through order $n-2$ vanish) and $p(1) = c = 0$; so the map is injective, hence bijective, and v_d is the preimage of d by the computation in Theorem 2.2(i). Given $p \in \mathbb{R}_{\leq n-1}[x]$ with $d = \mathrm{data}(p)$, uniqueness gives $p = v_d$, and Theorem 2.2(ii) says $d \in \mathcal{D}_n \iff p \in \mathcal{M}_{n-1}$. $\square$

So the forcing problem *is* the problem of describing, in coefficient space, the set of polynomials of bounded degree whose maximum on $[0,1]$ sits at an endpoint. That set has been met before, from a probabilistic angle: Bos and Ware [2] studied the probability that a random polynomial of degree at most n attains its maximum over a symmetric interval at an endpoint, with exact formulas in degrees 1 and 2 and an explicit lower bound in degree 3; Corollary 2.5 gives the same event a second reading, as the forcing of a maximum principle. The set splits into two convex pieces for trivial—but useful—reasons: the maximum is either at the left endpoint or at the right one.

Theorem 2.6 (Two regimes)**.** *Let*

$$\mathcal{A}_n = \big\{ d : v_d(x) \leq c_0 \ \ \textit{for all } x \in [0,1] \big\}, \qquad \mathcal{B}_n = \big\{ d : v_d(x) \leq \beta \ \ \textit{for all } x \in [0,1] \big\}.$$

Then $\mathcal{A}_n$ and $\mathcal{B}_n$ are closed convex cones, each an intersection of a one-parameter family of closed half-spaces, and

$$\mathcal{D}_n \ = \ \mathcal{A}_n \cup \mathcal{B}_n.$$

Proof. For fixed x, the map $d \mapsto v_d(x) - c_0$ is linear, so $\mathcal{A}_n = \bigcap_{x \in [0,1]} \{d : v_d(x) - c_0 \leq 0\}$ is a closed convex cone; likewise $\mathcal{B}_n$. If $d \in \mathcal{A}_n$ then $\max v_d = c_0 \leq \max\{c_0, \beta\}$, so $\mathcal{A}_n \subseteq \mathcal{D}_n$ by Theorem 2.2(ii); likewise $\mathcal{B}_n$. Conversely, if $d \in \mathcal{D}_n$ then $\max_{[0,1]} v_d = \max\{c_0, \beta\}$, and this benchmark is a single number, equal to c_0 or to β; accordingly $d \in \mathcal{A}_n$ or $d \in \mathcal{B}_n$. $\square$

Thus all failure of convexity in $\mathcal{D}_n$ comes from the union of two convex modes: "the maximum never leaves a" and "the maximum never leaves b". Theorem 5.2 will show that for $n \geq 4$ these two cones are genuinely curved.

Remark 2.7 (Effective tests). Membership in $\mathcal{D}_n$ is a finite computation: apply Sturm's algorithm to v_d' and compare critical values with $\max\{c_0, \beta\}$. More structurally, $\mathcal{A}_n$ is the preimage of the cone of polynomials of degree $\le n-1$ nonnegative on $[0,1]$ under the linear map $d \mapsto c_0 - v_d$, and by the Markov–Lukács representation of such polynomials (see [10, §1.21]) that cone is semidefinite-representable; so membership in $\mathcal{D}_n = \mathcal{A}_n \cup \mathcal{B}_n$ is decided by two small semidefinite programs. Contrast the definition of forcing, which quantifies over an infinite-dimensional class. Finally, $\mathcal{D}_n$ is semialgebraic, by Tarski–Seidenberg elimination [1] applied to the description in Theorem 2.2(ii).

Remark 2.8 (Theorem 1.1 revisited). In the present language, Theorem 1.1 asserts that the convex *polyhedral* cone

$$\Pi_n \;=\; \Big\{ d \,:\, \beta \ge \textstyle\sum_{i=0}^{j} \frac{c_i}{i!} \;\text{ for } j = 1, \dots, n-2 \Big\}$$

is contained in $\mathcal{D}_n$. By Corollary 2.5 this is a statement purely about polynomials of degree $\le n-1$—which explains why the induction in [7] runs entirely through the polynomials v_d. The next two sections locate Π_n inside the true forcing region: for $n=3$ the sharp threshold doubles the one in Π_3, and a second convex cone, invisible to Π_n, forces for every β.

3. The complete answer for $n = 3$

Write $t^+ = \max\{t, 0\}$.

Theorem 3.1. *Let $d = (c_0, c_1; \beta) \in \mathbb{R}^3$. Then*

$$d \in \mathcal{D}_3 \quad \Longleftrightarrow \quad c_1 \;\le\; 2\,(\beta - c_0)^+.$$

Un-normalized: for f with f'' nondecreasing on $[a,b]$ (in particular for $f \in C^3$, $f''' \ge 0$), the data $\big(f(a), f'(a); f(b)\big)$ is forcing if and only if

$$f'(a)\,(b-a) \;\le\; 2\,\max\{f(b) - f(a),\, 0\}.$$

Proof. By Theorem 2.2 we must decide when the quadratic

$$q(x) \;=\; v_d(x) \;=\; c_0 + c_1 x + A x^2, \qquad A = \beta - c_0 - c_1,$$

satisfies the maximum principle on $[0,1]$.

Case $A \ge 0$. Then q is convex, so the maximum principle holds. The stated inequality holds too: if $c_1 \le 0$ it is trivial, and if $c_1 > 0$ then $\beta - c_0 = A + c_1 \ge c_1 > 0$, so $2(\beta-c_0)^+ = 2(\beta - c_0) \ge 2c_1 > c_1$.

Case $A < 0$. Then q is strictly concave, and the maximum principle fails precisely when the vertex of q lies in the open interval $(0,1)$, i.e. when $q'(0) > 0 > q'(1)$. Now $q'(0) = c_1$, and since q is quadratic,

$$q'(0) + q'(1) \;=\; 2\big(q(1) - q(0)\big) \;=\; 2(\beta - c_0),$$

so $q'(1) < 0 \iff c_1 > 2(\beta - c_0)$. Thus failure is equivalent to $c_1 > 0$ *and* $c_1 > 2(\beta - c_0)$, i.e. to $c_1 > 2(\beta - c_0)^+$; forcing is equivalent to $c_1 \le 2(\beta - c_0)^+$. ☐

Remark 3.2 (Comparison with Theorem 1.1). For $n = 3$, Theorem 1.1 requires $\beta \ge c_0 + c_1$, i.e. $c_1 \le \beta - c_0$. Theorem 3.1 shows the sharp threshold is *twice* that: an initial slope up to twice the mean slope $\frac{f(b)-f(a)}{b-a}$ is still forcing (when the mean slope is positive), and any nonpositive initial slope is forcing unconditionally. The geometric meaning of the constant 2 is the reflection property of parabolas: for the extremal quadratic, slopes at the two endpoints average to the chord slope, so the vertex leaves the interval exactly when $f'(a)$ passes twice the chord slope.

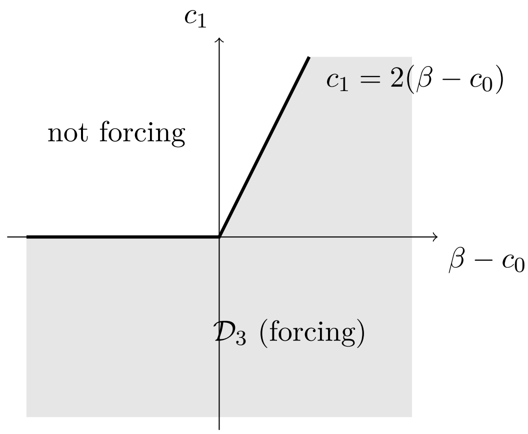

FIGURE 1. The forcing set $\mathcal{D}_3$ in the invariant coordinates $(\beta - c_0,\ c_1)$: the closed region below the kinked boundary $c_1 = 2(\beta - c_0)^+$ (the left half of the boundary runs along the horizontal axis). It is the union of the two closed half-spaces $\{c_1 \le 0\}$ and $\{c_1 \le 2(\beta - c_0)\}$.

Example 3.3. (i) The data $(0, 0; -1)$ of $-x^2$ on $[0, 1]$ satisfies $c_1 = 0 \le 2(-1)^+ = 0$: forcing, as promised in the introduction—the good behavior of $-x^2$ was no accident. (ii) The data $(0, 1; 0)$ of $x - x^2$ fails the criterion $(1 > 0)$, and indeed $x - x^2 = v_d$ is its own counterexample. (iii) For [7, Example 1.1], $f(x) = x^4 - 5x^2 + 4$ on $[0, 3]$: here $f'(0) = 0 \le 0$, so the data is forcing at a glance, with no computation at the right endpoint.

4. Descending data

The second convex cone inside $\mathcal{D}_n$ requires no condition on β at all. Its proof does not use Theorem 2.2; it is an elementary sign-change argument, valid directly on $[a, b]$.

Theorem 4.1 (Descending data)**.** *Let $n \ge 2$ and let $f \in C^{n-1}([a, b])$ with $f^{(n-1)}$ nondecreasing. If*

$$f^{(i)}(a) \ \le\ 0 \qquad \text{for } i = 1, \dots, n-2,$$

then there is $c \in [a, b]$ such that f is nonincreasing on $[a, c]$ and nondecreasing on $[c, b]$. In particular $f \le \max\{f(a), f(b)\}$, so

$$\big\{d \in \mathbb{R}^n : c_1 \le 0, \dots, c_{n-2} \le 0\big\} \ \subseteq\ \mathcal{D}_n \qquad \text{for every value of } \beta.$$

Proof. Say that $h \in C([a, b])$ has property (S) if there is $\tau \in [a, b]$ with $h \le 0$ on $[a, \tau)$ and $h \ge 0$ on $(\tau, b]$ (“at most one sign change, from $-$ to $+$”).

Step 1: every nondecreasing h has (S). The set $\{h < 0\}$ is an initial segment of $[a, b]$; take τ to be its supremum (or $\tau = a$ if it is empty).

Step 2: if $H \in C^1([a, b])$ and H' has (S) with split point τ, then H is nonincreasing on $[a, \tau]$ and nondecreasing on $[\tau, b]$; if moreover $H(a) \le 0$, then H has (S). The monotonicity is immediate from the sign of H' on the two subintervals. Assuming $H(a) \le 0$: on $[a, \tau]$ we get $H \le H(a) \le 0$; on $[\tau, b]$, H is nondecreasing, so $\{H < 0\} \cap [\tau, b]$ is an initial segment of $[\tau, b]$, and its supremum τ' splits $[a, b]$ as required.

Now $f^{(n-1)}$ is nondecreasing, hence has (S) by Step 1. Applying the second half of Step 2 successively to $H = f^{(i)}$ for $i = n-2, n-3, \dots, 1$—using the hypothesis $f^{(i)}(a) \le 0$ at each stage—we conclude that each $f^{(i)}$ has (S), down to f'. A final application of the first half of Step 2 to $H = f$ (no sign condition on $f(a)$ needed) yields the claimed V-shape, and a V-shaped function attains its maximum at an endpoint. $\square$

Remark 4.2. The descending cone and Pan's cone Π_n are incomparable: descending data with β very negative violates every condition $\beta \geq \sum_{i\leq j} c_i/i!$ with c_0 fixed (take $d = (0, 0, \ldots, 0; -1)$, which is descending but not in Π_n), while Π_n allows $c_1 > 0$. For $n = 3$, Theorem 3.1 exhibits $\mathcal{D}_3$ as exactly the union of the descending half-space $\{c_1 \leq 0\}$ and the half-space $\{c_1 \leq 2(\beta - c_0)\}$ containing Π_3—the two-cone structure of Theorem 2.6 in explicit, and here even polyhedral, form. Note also that neither cone alone suffices: the data $(0, 1; 0.6)$ is forcing $(1 \leq 1.2)$ but lies in neither $\{c_1 \leq 0\}$ nor Π_3.

5. An explicit slice, and the geometry of $\mathcal{D}_n$

For $n \geq 4$ we now compute a two-dimensional affine slice of $\mathcal{D}_n$ in closed form. It displays, side by side, the linear and the genuinely curved parts of the boundary.

Proposition 5.1 (The slice)**.** *Let $n \geq 4$. For $(s, u) \in \mathbb{R}^2$ define $\iota(s, u) \in \mathbb{R}^n$ by*

$$\iota(s,u) \;=\; \begin{cases} \big(0,\ s,\ u;\ s + \frac{u}{2} - 1\big), & n = 4, \\ \big(0,\ s,\ u,\ -6,\ 0, \ldots, 0;\ s + \frac{u}{2} - 1\big), & n \geq 5, \end{cases}$$

i.e. $c_0 = 0$, $c_1 = s$, $c_2 = u$, $c_3 = -6$ and $c_i = 0$ for $4 \leq i \leq n-2$ when present, and $\beta = s + \frac{u}{2} - 1$. Then for every $n \geq 4$ the extremal polynomial is the same cubic,

$$v_{\iota(s,u)}(x) \;=\; w(x) \;=\; s\,x + \tfrac{u}{2}\,x^2 - x^3,$$

and

$$\iota(s,u) \in \mathcal{D}_n \quad\iff\quad u \geq 4 \quad or \quad s + u \geq 3 \quad or \quad s \leq -\tfrac{1}{16}\,(u^+)^2.$$

Proof. The extremal polynomial. For $n = 4$: $T_d = sx + \frac{u}{2}x^2$ and $\beta - T_d(1) = (s + \frac{u}{2} - 1) - (s + \frac{u}{2}) = -1$, so $v_d = T_d - x^3 = w$. For $n \geq 5$: the entry $c_3 = -6$ contributes $\frac{c_3}{3!}x^3 = -x^3$, so $T_d = w$ has degree $3 \leq n - 2$, and $\beta - T_d(1) = 0$; again $v_d = w$.

By Theorem 2.2, $\iota(s, u) \in \mathcal{D}_n$ iff $w \leq \max\{0, \beta\}$ on $[0, 1]$, where $\beta = w(1)$.

Shape analysis. The derivative $w'(x) = s + ux - 3x^2$ is a concave quadratic, so $\{w' \geq 0\}$ is a closed interval $[r, \sigma]$ (possibly empty), and on $[0, 1]$ the function w is nonincreasing, then nondecreasing, then nonincreasing. We record the key identity: if $w'(\sigma) = 0$, then $s = 3\sigma^2 - u\sigma$, whence

$$w(\sigma) \;=\; (3\sigma^2 - u\sigma)\sigma + \tfrac{u}{2}\sigma^2 - \sigma^3 \;=\; \tfrac{\sigma^2}{2}\,\big(4\sigma - u\big). \tag{1}$$

Case 1: $w'(1) \geq 0$, i.e. $s + u \geq 3$. Then $1 \in [r, \sigma]$, so on $[0, 1]$ the function w is nonincreasing then nondecreasing; its maximum is at an endpoint, and the data is forcing.

Case 2: $w'(1) < 0$ and $w' \leq 0$ on $[0, 1]$. Then w is nonincreasing, $\max_{[0,1]} w = w(0) = 0 = \max\{0, \beta\}$ (as $\beta = w(1) \leq 0$): forcing.

Case 3: $w'(1) < 0$ and $w'(x_0) > 0$ for some $x_0 \in (0, 1)$. Then $r < \sigma$ are real and $x_0 \in (r, \sigma)$; since $w'(1) < 0$ excludes $1 \in [r, \sigma]$, we get $0 < x_0 < \sigma < 1$. So w increases up to σ and decreases on $[\sigma, 1]$; hence

$$\max_{[0,1]} w \;=\; \max\{w(0), w(\sigma)\} \;=\; \max\{0, w(\sigma)\}, \qquad \beta = w(1) < w(\sigma).$$

If $w(\sigma) \leq 0$ the maximum is $0 \leq \max\{0, \beta\}$: forcing. If $w(\sigma) > 0$, then $\max w = w(\sigma)$ exceeds both 0 and β: not forcing. By (1) and $\sigma > 0$,

$$\text{forcing} \iff w(\sigma) \leq 0 \iff \sigma \leq \tfrac{u}{4}.$$

Assembly. Suppose $u \geq 4$. Cases 1 and 2 force; in Case 3, $\sigma < 1 \leq \frac{u}{4}$, so it forces as well. Thus $u \geq 4$ always forces.

Suppose $u < 4$ and $s + u < 3$ (so Case 1 is excluded); we claim forcing $\iff$ $s \leq -\frac{1}{16}(u^+)^2$.

If $u \le 0$: for $s \le 0$ we have $w'(x) = s + ux - 3x^2 \le s \le 0$ on $[0,1]$, Case 2, forcing; for $s > 0$ we have $w' > 0$ near 0, Case 3, with $\sigma > 0 \ge \frac{u}{4}$, hence $w(\sigma) > 0$: not forcing. This matches the claim, whose right-hand side is 0 for $u \le 0$.

If $0 < u < 4$: when $u^2 + 12s < 0$, w' has no real root and is negative, Case 2, forcing—and indeed $s < -\frac{u^2}{12} < -\frac{u^2}{16}$. When $u^2 + 12s = 0$, $w' \le 0$, Case 2, forcing, and $s = -\frac{u^2}{12} \le -\frac{u^2}{16}$. When $u^2 + 12s > 0$, the roots are

$$r = \tfrac{1}{6}\big(u - \sqrt{u^2 + 12s}\big) \;<\; \sigma = \tfrac{1}{6}\big(u + \sqrt{u^2 + 12s}\big),$$

with $\sigma \ge \frac{u}{6} > 0$ and $r \le \frac{u}{6} < 1$, so $(r,\sigma) \cap (0,1) \ne \emptyset$ and we are in Case 3. Then

$$\text{forcing} \iff \sigma \le \tfrac{u}{4} \iff \sqrt{u^2 + 12s} \le \tfrac{u}{2} \iff s \le -\tfrac{u^2}{16}. \qquad \square$$

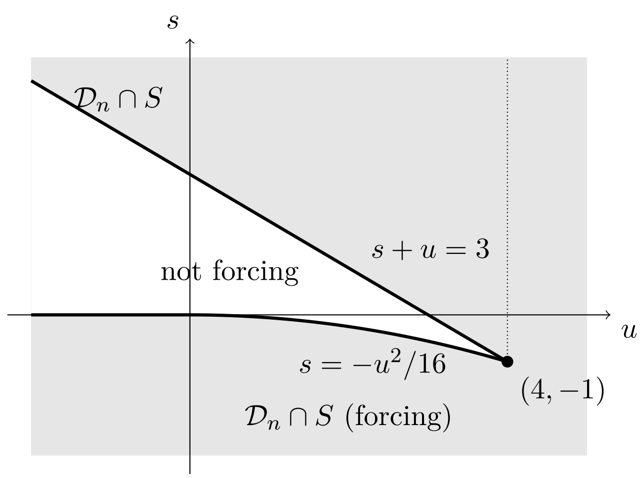

FIGURE 2. The slice $S = \iota(\mathbb{R}^2)$ of Proposition 5.1, drawn in the coordinates $(u, s) = (c_2, c_1)$, valid for every $n \ge 4$. The non-forcing set is the open lens between the line $s + u = 3$ and the parabolic arc $s = -u^2/16$, pinching off at $(4, -1)$; to the right of the dotted line $u = 4$ everything is forcing. The curved arc is the image of the double-contact cubics $w = -x(x - u/4)^2$, for which the interior local maximum exactly reaches the benchmark 0. No such curvature is possible for $n = 3$ (Figure 1).

The boundary arc deserves a name: for $0 < m < 1$ the point $(s, u) = (-m^2, 4m)$ lies on the parabola, and there the extremal cubic is

$$w(x) \;=\; -x\,(x - m)^2,$$

the *double-contact* configuration: $w \le 0$ on $[0,1]$ with equality at the endpoint 0 and, tangentially, at the interior point m. These tangential extremals are exactly what a finite list of linear conditions cannot see.

Theorem 5.2 (Geometry of the forcing cone)**.** *Let $n \ge 3$.*

(a) *$\mathcal{D}_n$ is a closed cone: $\lambda \mathcal{D}_n \subseteq \mathcal{D}_n$ for $\lambda \ge 0$, and $\mathcal{D}_n$ is invariant under $d \mapsto d + t\,(1, 0, \dots, 0; 1)$, $t \in \mathbb{R}$ (adding constants to f).*
(b) *$\mathcal{D}_3 = \{c_1 \le 0\} \cup \{c_1 \le 2(\beta - c_0)\}$ is a union of two closed half-spaces.*
(c) *$\mathcal{D}_n$ is not convex.*
(d) *For $n \ge 4$, $\mathcal{D}_n$ is not a finite union of convex polyhedra. In particular, no finite list of linear inequalities in the data, applied case-wise, decides forcing once $n \ge 4$.*

Proof. (a) For $\lambda \geq 0$, $v_{\lambda d} = \lambda v_d$ and both sides of the criterion in Theorem 2.2(ii) scale by λ. For the shift $e = (1, 0, \ldots, 0; 1)$, we get $T_{d+te} = T_d + t$ and $\beta + t - T_{d+te}(1) = \beta - T_d(1)$, so $v_{d+te} = v_d + t$, while the benchmark also shifts by t. Closedness: $\Phi(d) = \max_{x\in[0,1]} v_d(x) - \max\{c_0, \beta\}$ is continuous (a maximum over a compact set of a function jointly continuous in (d, x)), and $\mathcal{D}_n = \Phi^{-1}\big((-\infty, 0]\big)$.

(b) If $c_1 \leq 2(\beta - c_0)^+$ then either $c_1 \leq 0$, or $c_1 > 0$, in which case $(\beta - c_0)^+ = \beta - c_0$ and $c_1 \leq 2(\beta - c_0)$. Conversely, $\{c_1 \leq 0\} \subseteq \mathcal{D}_3$ trivially from Theorem 3.1, and if $c_1 \leq 2(\beta - c_0)$ then either $\beta \geq c_0$, giving $c_1 \leq 2(\beta - c_0)^+$, or $\beta < c_0$, in which case $c_1 \leq 2(\beta - c_0) < 0 \leq 2(\beta - c_0)^+$.

(c) Consider (all unlisted c_i equal to 0)

$$d_- = (0, -1, 0, \ldots, 0;\ -10), \quad d_+ = (0, 11, 0, \ldots, 0;\ 11), \quad d_0 = \tfrac{1}{2}(d_- + d_+) = (0, 5, 0, \ldots, 0;\ \tfrac{1}{2}).$$

d_- is forcing by Theorem 4.1; d_+ is forcing because $v_{d_+}(x) = 11x + (11 - 11)x^{n-1} = 11x$ has its maximum $11 = \beta$ at $x = 1$. But $v_{d_0}(x) = 5x - \frac{9}{2}x^{n-1}$ satisfies

$$v_{d_0}\big(\tfrac{1}{2}\big) \;=\; \tfrac{5}{2} - \tfrac{9}{2}\cdot 2^{1-n} \;\geq\; \tfrac{5}{2} - \tfrac{9}{8} \;=\; \tfrac{11}{8} \;>\; \tfrac{1}{2} \;=\; \max\{c_0, \beta\},$$

so $d_0 \notin \mathcal{D}_n$ by Theorem 2.2(ii).

(d) Suppose $\mathcal{D}_n = Q_1 \cup \cdots \cup Q_N$ with each Q_i a convex polyhedron (a finite intersection of closed half-spaces). Intersect with the affine 2-plane $S = \iota(\mathbb{R}^2)$ of Proposition 5.1: each $Q_i \cap S$ is a polyhedron in S, so its relative boundary lies on finitely many affine lines of S; since $\partial\big(\bigcup A_i\big) \subseteq \bigcup \partial A_i$, the relative boundary of $\mathcal{D}_n \cap S$ lies on finitely many lines. But consider the parabolic arc

$$\Gamma \;=\; \big\{(s, u) = (-m^2,\, 4m) \;:\; 0 < m < 1\big\}.$$

Each point of Γ lies in $\mathcal{D}_n \cap S$ (third clause of Proposition 5.1, with equality), while for small $\varepsilon > 0$ the point $(-m^2 + \varepsilon, 4m)$ fails all three clauses: $-m^2 + \varepsilon > -\frac{(4m)^2}{16}$; $4m < 4$; and $s + u = -m^2 + 4m + \varepsilon < 3$ for $\varepsilon < (1 - m)(3 - m)$. Hence Γ lies in the relative boundary of $\mathcal{D}_n \cap S$. A line meets the parabola $s = -u^2/16$ in at most two points, while Γ is infinite—a contradiction. ☐

Remark 5.3. Combining Theorems 2.6 and 5.2(d): $\mathcal{D}_n$ is always a union of *two* closed convex cones, but for $n \geq 4$ those cones cannot be chosen polyhedral—a union of finitely many polyhedral cones is a finite union of polyhedra. For $n = 3$ they can: Theorem 5.2(b). The dichotomy is thus not about the number of convex pieces, which is always two, but about their curvature.

Remark 5.4 (Connectivity). $\mathcal{D}_n$ is star-shaped about the origin, hence connected (indeed contractible). Its complement is also path-connected for $n \geq 3$: working in the polynomial model $\mathcal{M}_{n-1}$ (Corollary 2.5), let $B(x) = x(1 - x)$, which has degree $2 \leq n - 1$ and vanishes at the endpoints. If $p \notin \mathcal{M}_{n-1}$, with witness $p(x_*) > \max\{p(0), p(1)\}$, then $p + tB \notin \mathcal{M}_{n-1}$ for all $t \geq 0$, since the endpoint values are unchanged while the value at x_* only grows. Given two non-members p, q, slide each to $p + TB$, $q + TB$ and join by the segment $(1-\lambda)p + \lambda q + TB$: its endpoint values are bounded by $E = \max\{p(0), p(1), q(0), q(1)\}$, while its value at $\frac{1}{2}$ is at least $\frac{T}{4} - C$ with $C = \max_{[0,1]}(|p| + |q|)$; for $T > 4(C + E)$ the whole path stays outside $\mathcal{M}_{n-1}$. So the failure of convexity of $\mathcal{D}_n$ is a single curved bite out of a cone, not fragmentation.

Impossibility of a flat description is not impossibility of a description. For $n = 4$ the two invariances of Theorem 5.2(a) reduce the four-dimensional data space to a two-dimensional quotient, and Proposition 5.1 was secretly not a sample of $\mathcal{D}_4$ but an entire chart of it: the slice $\{c_0 = 0,\ \beta = c_1 + \frac{c_2}{2} - 1\}$ consists exactly of the normalized data whose extremal cubic has leading coefficient -1. What is missing is the mirror chart, and a reflection supplies it.

Lemma 5.5 (Reflection)**.** *The involution $p \mapsto \check p$, $\check p(x) = p(1-x)$, maps $\mathcal{M}_k$ onto itself. In the charts of Proposition 5.1: if $p(x) = s\,x + \frac{u}{2}x^2 - x^3$ up to an additive constant, then $\check p(x) = s'x + \frac{u'}{2}x^2 + x^3$ up to an additive constant, with*

$$(s', u') \;=\; (3 - s - u,\; u - 6).$$

Proof. $\max_{[0,1]} \check p = \max_{[0,1]} p$ and $\{\check p(0), \check p(1)\} = \{p(1), p(0)\}$, so $p \in \mathcal{M}_k \iff \check p \in \mathcal{M}_k$. Expanding $p(1-x) = s(1-x) + \frac{u}{2}(1-x)^2 - (1-x)^3$ gives constant term $s + \frac{u}{2} - 1$ and then $(3-s-u)\,x + (\frac{u}{2} - 3)\,x^2 + x^3$. □

Theorem 5.6 (The forcing set for $n = 4$)**.** *Let $d = (c_0, c_1, c_2; \beta) \in \mathbb{R}^4$ and let $A = \beta - c_0 - c_1 - \frac{c_2}{2}$ be the leading coefficient of the extremal cubic v_d. Then $d \in \mathcal{D}_4$ if and only if the case matching the sign of A holds:*

(0) *$A = 0$: $\quad c_1 \le 2\,(\beta - c_0)^+$;*

$(-)$ *$A < 0$: $\quad c_2 \ge 4|A|$, or $c_1 + c_2 \ge 3|A|$, or $16\,|A|\,c_1 \le -\big(c_2^+\big)^2$;*

$(+)$ *$A > 0$: $\quad c_1 \le 0$, or $c_2 \ge -2A$, or $\big[c_2 \le -6A$ and $c_1 + c_2 + 3A \ge 0\big]$, or $\big[-6A \le c_2 \le -2A$ and $16\,A\,c_1 \ge (c_2 - 6A)(c_2 + 2A)\big]$.*

Proof. Case $A = 0$. Here v_d is the quadratic $c_0 + c_1 x + \frac{c_2}{2}x^2$ and the argument of Theorem 3.1 applies verbatim: if $c_2 \ge 0$ the quadratic is convex, the maximum principle holds, and the stated inequality holds automatically (if $c_1 > 0$ then $\beta - c_0 = c_1 + \frac{c_2}{2} \ge c_1 > 0$); if $c_2 < 0$ the principle fails precisely when $v_d'(0) > 0 > v_d'(1)$, and $v_d'(0) + v_d'(1) = 2(\beta - c_0)$ turns this into $c_1 > 2(\beta - c_0)^+$.

Case $A < 0$. The invariances of Theorem 5.2(a) carry d to data with $c_0 = 0$ and leading coefficient -1, that is, into the slice of Proposition 5.1 (with $n = 4$), at chart coordinates $(s, u) = (c_1/|A|,\, c_2/|A|)$. The three clauses of $(-)$ are precisely the homogenizations of $u \ge 4$, $s + u \ge 3$, and $s \le -\frac{1}{16}(u^+)^2$.

Case $A > 0$. The extremal cubic of the reflected data $\check d := \mathrm{data}\big(v_d(1-\cdot)\big)$ is $v_d(1-\cdot)$ itself (both are cubics with the same four Hermite conditions, so they coincide by the uniqueness in Corollary 2.5), and its leading coefficient is $-A < 0$; by Lemma 5.5, $d \in \mathcal{D}_4 \iff \check d \in \mathcal{D}_4$, which case $(-)$ decides. It remains to transport the answer. Write $(s', u') = (c_1/A,\, c_2/A)$ for the chart coordinates of the normalization of d, and (s, u) for those of its reflection; inverting the chart map of Lemma 5.5 gives

$$u = u' + 6, \qquad s = 3 - s' - u = -3 - s' - u'.$$

Under this substitution the non-forcing lens of Proposition 5.1, $\{-\frac{1}{16}(u^+)^2 < s < 3 - u,\ u < 4\}$, pulls back clause by clause: $u < 4 \iff u' < -2$; the constraint $s < 3 - u = -3 - u'$ becomes $s' > 0$; and the constraint $s > -\frac{1}{16}(u^+)^2$ becomes

$$s' \;<\; \begin{cases} -u' - 3, & u' \le -6 \quad (\text{i.e. } u \le 0,\ \text{where } (u^+)^2 = 0),\\ \frac{1}{16}(u' - 6)(u' + 2), & -6 \le u' < -2 \quad (\text{i.e. } 0 \le u < 4),\end{cases}$$

using $\frac{1}{16}(u'+6)^2 - 3 - u' = \frac{1}{16}\big(u'^2 - 4u' - 12\big) = \frac{1}{16}(u'-6)(u'+2)$. So for $A > 0$ the data fails to force exactly when $s' > 0$, $u' < -2$, and s' lies below the two-piece bound above; un-normalizing $(s', u') = (c_1/A, c_2/A)$ yields the negation of the clauses in $(+)$. □

Remark 5.7. Geometrically: the non-forcing set in $\mathbb{R}^4$ consists of two curved lenses, mirror images of one another under $x \mapsto 1 - x$, one on each side of the hyperplane $A = 0$ (Figure 2 shows the $A < 0$ lens). They are glued along that hyperplane: as $A \to 0^\pm$ both lenses converge onto the strip $\{0 < c_1 < -c_2\}$, which is exactly the non-forcing set of case (0), since there $\beta - c_0 = c_1 + \frac{c_2}{2}$ and $c_1 > 2(\beta - c_0)^+ \iff 0 < c_1 < -c_2$. This completes the description problem for $n = 4$;

Question 6.5 below records what remains for $n \geq 5$. In particular, combining Theorem 5.6 with an explicit integration would in principle upgrade the degree-3 endpoint-maximum probability bound of [2] to an exact value.

6. Complements and questions

Remark 6.1 (Data at both endpoints: a parity phenomenon). The choice of data—$n-1$ derivatives at a and one value at b—is not as arbitrary as it looks. First, the count: the n conditions match the rigidity of $\mathbb{R}_{\leq n-1}[x]$, and this is what Theorem 2.2 exploits—drop any one condition and the enlarged class no longer has a greatest element. Indeed, if the value $f^{(j)}(a)$ is freed for some $1 \leq j \leq n-2$, the polynomial

$$\varphi(x) \;=\; (x-a)^j\Big((b-a)^{n-1-j} - (x-a)^{n-1-j}\Big)$$

is nonnegative on $[a,b]$, positive on (a,b), of degree $n-1$, and vanishes at both endpoints together with every retained derivative at a; so $f + t\varphi$ remains in the class for all $t \geq 0$ and is unbounded at interior points. (If instead $f(a)$ is freed, the $j=0$ member of the same family works: $t\big((b-a)^{n-1} - (x-a)^{n-1}\big)$; if $f(b)$ is freed, use $t\,(x-a)^{n-1}$. Both preserve every other condition.) Whether *forcing* survives is a finer question, because the benchmark $\max\{f(a), f(b)\}$ moves with f. If an interior derivative $f^{(j)}(a)$ is freed while both endpoint values are retained, the benchmark stays put under $f \mapsto f + t\varphi$ while interior values blow up: no such partial data forces. But the endpoint values themselves are dispensable: Theorem 4.1 forces with the $n-2$ conditions $f^{(i)}(a) \leq 0$ alone, both $f(a)$ and $f(b)$ left free. The interior derivative slots are indispensable; the endpoint values are not.

Second, distributing the n conditions differently over the two endpoints produces a parity dichotomy. Fix $1 \leq k \leq n-1$ and prescribe

$$f^{(i)}(a)\ (0 \leq i \leq n-1-k), \qquad f^{(j)}(b)\ (0 \leq j \leq k-1),$$

n conditions in all, and take $f \in C^n$, $f^{(n)} \geq 0$. The unique Hermite interpolant $H_d \in \mathbb{R}_{\leq n-1}[x]$ to this data satisfies (see [4, Chapter 3])

$$f(x) - H_d(x) \;=\; \frac{f^{(n)}(\xi_x)}{n!}\,(x-a)^{n-k}(x-b)^k,$$

whose sign on $[a,b]$ is $(-1)^k$. For *odd* k, H_d is again the greatest element of the class, the entire analysis of this paper applies verbatim, and by the analogue of Corollary 2.5 the forcing set for the split k is the image of the *same* cone $\mathcal{M}_{n-1}$ under a linear isomorphism: Theorems 2.6 and 5.2 transfer unchanged, and all odd-k forcing sets are linearly isomorphic to one another. For *even* k, H_d is instead the *least* element, the supremum over the class is $+\infty$ at interior points (add $t\,(x-a)^{n-k}(x-b)^k$), so *no* data forces the maximum principle—but every statement dualizes to the minimum principle $f \geq \min\{f(a), f(b)\}$. The 2013 note is the case $k=1$.

For $n \geq 5$ the moduli of data is $(n-2)$-dimensional and we do not know $\mathcal{D}_n$ explicitly. But the boundary structure suggested by Theorem 5.6 can be pinned down in every degree.

Theorem 6.2 (The boundary of $\mathcal{M}_k$)**.** *Let $k \geq 2$, $p \in \mathcal{M}_k$, and $B = \max\{p(0), p(1)\}$. Then $p \in \partial\mathcal{M}_k$ if and only if*

(T) *$p(m) = B$ for some $m \in (0,1)$ (such an m is automatically a critical point: an interior tangency); or*

(E) *some endpoint at which B is attained is a critical point of p.*

Consequently $\partial\mathcal{M}_k$ is contained in the union of four algebraic hypersurfaces: the hyperplanes $\{p'(0)=0\}$, $\{p'(1)=0\}$, and the two tangency hypersurfaces

$$\Big\{\,\mathrm{Res}_x\Big(p'(x),\ \tfrac{p(x)-p(j)}{x-j}\Big)=0\,\Big\},\qquad j\in\{0,1\},$$

whose defining polynomials have degree at most $2k-2$ in the coefficients. Moreover a boundary point carries at most $\lfloor (k-1)/2\rfloor$ simultaneous interior tangencies, and this is sharp for every $k\ge 3$; the first double tangencies occur at $k=5$, e.g.

$$p(x)\ =\ m_1^2m_2^2\ -\ (1-x)\,(x-m_1)^2(x-m_2)^2,\qquad 0<m_1<m_2<1.$$

Proof. ($\Leftarrow$) If (T) holds, then for every $\varepsilon>0$ the polynomial $p+\varepsilon\,x(1-x)$ has the same endpoint values as p but exceeds B at m; so p is a limit of non-members. If (E) holds, say at 0: then $p(0)=B$, $p'(0)=0$, so $B-p(x)\le Cx^2$ on $[0,1]$ for some $C>0$, and $q_\varepsilon=p+\varepsilon\,x(1-x)$ satisfies, at $\delta=\min\{\varepsilon/(4C),\frac12\}$,

$$q_\varepsilon(\delta)-B\ \ge\ \varepsilon\,\delta(1-\delta)-C\delta^2\ \ge\ \tfrac{\varepsilon\delta}{2}-C\delta^2\ >\ 0,$$

again with unchanged endpoint values: $q_\varepsilon\notin\mathcal{M}_k$. The endpoint 1 is symmetric.

($\Rightarrow$) Suppose neither (T) nor (E) holds; we show every q with $\|q-p\|_{C^1[0,1]}$ small lies in $\mathcal{M}_k$ (coefficientwise-close implies C^1-close, so p is then interior). If $p(0)=B$, then $p'(0)<0$ (the sign forced by $p\le B$, the strictness by the failure of (E)), so $p'\le -2\gamma$ on $[0,2\delta]$ for some $\gamma,\delta>0$; every nearby q is decreasing there, whence $q\le q(0)\le\max\{q(0),q(1)\}$ on $[0,2\delta]$. If instead $p(0)<B$, then $p\le B-2\gamma$ on $[0,2\delta]$ by continuity, and nearby q satisfy $q\le B-\gamma\le\max\{q(0),q(1)\}$ there, since the benchmark of q moves by at most $\|q-p\|_\infty$. The endpoint 1 is symmetric, and on $[\delta,1-\delta]$ the failure of (T) gives $p\le B-2\gamma'$, so the same comparison applies. Hence $q\in\mathcal{M}_k$.

For the containment: an interior tangency at level $B=p(j)$ makes $m\in(0,1)$ a common root of $p'(x)$ and of $(p(x)-p(j))/(x-j)$, a polynomial of degree $k-1$ that vanishes at m because $p(m)=p(j)$ and $m\ne j$; so the resultant of these two degree-$(k-1)$ polynomials vanishes, and since their coefficients are affine in the a_i, it has degree at most $2k-2$. Each interior tangency contributes a double root of $p-B$ and the dominant endpoint a root as well, so τ tangencies force $2\tau+1\le k$, i.e. $\tau\le\lfloor(k-1)/2\rfloor$. Sharpness: for odd $k=2\tau+1$ take $p=B-(1-x)\prod_{i=1}^{\tau}(x-m_i)^2$ with $B=\prod_i m_i^2$ (displayed above for $\tau=2$): it lies in $\mathcal{M}_k$ with benchmark $B=p(1)>p(0)=0$ and ties at every m_i. For even k insert the extra factor $(x+1)$, positive on $[0,1]$, into the product. ☐

So the *algebraic* complexity of $\partial\mathcal{M}_k$ grows only linearly with k: four families of equations, of degrees $1,1,\le 2k-2,\le 2k-2$. The finer question is the chamber structure—which pieces of the four hypersurfaces actually bound, and how they fit together. This too can be organized. For nonconstant $p\in\partial\mathcal{M}_k$ with benchmark $B=\max\{p(0),p(1)\}$, define the *contact type* of p to be the triple $(\mu_0,\mu_1;\vec{\nu})$, where $\mu_j\ge 0$ is the multiplicity of $B-p$ at the endpoint j (zero when the endpoint is not dominant, so $\max(\mu_0,\mu_1)\ge 1$), and $\vec{\nu}=(\nu_1,\dots,\nu_\tau)$ lists the multiplicities of the interior zeros of $B-p$ in increasing order of position—each ν_i even, since $B-p\ge 0$. In this language, Theorem 6.2 says that a nonconstant $p\in\mathcal{M}_k$ is a boundary point precisely when its type has $\tau\ge 1$ or $\max(\mu_0,\mu_1)\ge 2$; call such a type, with total multiplicity $M:=\mu_0+\mu_1+\sum_i\nu_i\le k$, *admissible.*

Theorem 6.3 (The boundary poset)**.** *Fix $k\ge 3$, and identify $\mathbb{R}_{\le k}[x]\cong\mathbb{R}^{k+1}$.*

(a) *Every admissible type is realized on $\partial\mathcal{M}_k$, and the set of its boundary points is connected, of dimension $k+2+\tau-M$.*

(b) *Exactly four admissible types have codimension one: the endpoint walls E_0, E_1 ($\tau = 0$, one dominant endpoint of multiplicity 2) and the tangency walls T_0, T_1 ($\tau = 1$, $\nu_1 = 2$, one dominant endpoint of multiplicity 1). Thus each of the four hypersurfaces of Theorem 6.2 bounds $\mathcal{M}_k$ along a single connected piece: the chamber structure does not fragment.*

(c) *The number of admissible types with all multiplicities ≤ 2 is $4k-6$; the total number of admissible types grows exponentially: it lies between $2^{\lfloor (k-1)/2 \rfloor - 1}$ and $(k+1)^2\, 2^{k/2}$.*

Proof. (a) A boundary point p of the given type determines a unique factorization

$$B - p \;=\; x^{\mu_0}\,(1-x)^{\mu_1} \prod_{i=1}^{\tau} (x - m_i)^{\nu_i}\; h, \qquad 0 < m_1 < \cdots < m_\tau < 1,$$

where $h \in \mathbb{R}_{\leq k-M}[x]$ has no zeros in $[0,1]$ and is positive there (the explicit factors are nonnegative on $[0,1]$, and $B - p \geq 0$). Write $F(x) = x^{\mu_0}(1-x)^{\mu_1} \prod_i (x - m_i)^{\nu_i}$ for the contact factor, so that $B - p = Fh$. Conversely, every choice of $B \in \mathbb{R}$, of ordered positions, and of h positive on $[0,1]$ with $\deg h \leq k - M$ produces a member $p := B - Fh$ of $\mathcal{M}_k$ with exactly this type—$p \leq B$ with equality on the contact set, and the benchmark is B because $\max(\mu_0, \mu_1) \geq 1$—which is a boundary point by Theorem 6.2 whenever the type is admissible. The parameter set is the product of a line, an open simplex, and the convex cone of polynomials positive on $[0,1]$ of degree $\leq k - M$; it is connected, and the assignment is a bijection onto the stratum. It is moreover a homeomorphism: the inverse recovers B as the larger endpoint value, the contact positions as the zeros of $B - p$ in $(0,1)$ (which, being separated for fixed type, vary continuously with p), and the cofactor as $h = (B - p)/F$. Hence the stratum is nonempty, connected, and of dimension $1 + \tau + (k - M + 1)$.

(b) Codimension one means $M = \tau + 2$. Since each $\nu_i \geq 2$, we have $M \geq 2\tau + 1$, so $\tau \leq 1$. If $\tau = 0$, then $M = 2$ and admissibility rules out $(\mu_0, \mu_1) = (1,1)$: one dominant endpoint of multiplicity 2, giving E_0 or E_1. If $\tau = 1$, then $M = 3$, $\nu_1 = 2$, and one endpoint has multiplicity 1: T_0 or T_1.

(c) With all multiplicities ≤ 2: for the left-dominant types ($\mu_1 = 0$), $\mu_0 = 1$ admits $\tau \in \{1, \ldots, \lfloor (k-1)/2 \rfloor\}$ and $\mu_0 = 2$ admits $\tau \in \{0, \ldots, \lfloor (k-2)/2 \rfloor\}$, together $k-1$ types; symmetrically $k-1$ right-dominant; and the four both-dominant combinations $(\mu_0, \mu_1) \in \{1,2\}^2$ contribute $\lfloor \frac{k-2}{2} \rfloor + 2\lfloor \frac{k-1}{2} \rfloor + \lfloor \frac{k-2}{2} \rfloor = 2(k-2)$; in total $4k - 6$. For the full count, the interior data ranges over compositions into even parts: there are 2^{j-1} compositions of $2j$, so already the types with $(\mu_0, \mu_1) = (1,0)$ number at least $2^{\lfloor (k-1)/2 \rfloor - 1}$, while $(k+1)^2\, 2^{k/2}$ bounds all choices from above. ☐

The incidence structure of these strata—which degenerations glue which—is also combinatorial.

Theorem 6.4 (Closure relations)**.** *Let T, T' be admissible types, with strata $S(T), S(T') \subseteq \partial\mathcal{M}_k$. Then $S(T') \subseteq \overline{S(T)}$ if and only if T' is obtained from T by finitely many elementary degenerations:*

(i) *merging two adjacent interior contacts (adding their multiplicities);*
(ii) *absorbing the leftmost (rightmost) interior contact into the left (right) wall;*
(iii) *raising an interior multiplicity by an even amount, raising an endpoint multiplicity by any amount (possibly from zero), or inserting a new interior contact of even multiplicity;*

subject only to the budget $M' \leq k$. Moreover every closure $\overline{S(T)}$ contains the line of constants.

Proof. Finite sequences of the moves (i)–(iii) compose into a single certificate: a weakly monotone assignment of the interior contacts of T to the slots of T' (several to one interior slot; extreme ones possibly to the walls), together with multiplicity increments $e_L, e_R \geq 0$ at the walls and even increments $e_j \geq 0$ at the interior slots—and conversely. We work with certificates.

($\supseteq$) Fix a target $p' \in S(T')$ with factorization datum $(B', \vec{m}\,', h')$ as in Theorem 6.3(a). Set

$$h^* \;=\; x^{e_L}\,(1-x)^{e_R} \prod\nolimits_j (x-m_j')^{e_j}\; h',$$

of degree $\leq (M'-M)+(k-M') = k-M$. For $t \in [0,1)$ let $p_t \in S(T)$ have benchmark B', cofactor $h_t := h^* + (1-t)$, and interior contacts at distinct ordered positions converging, as $t \to 1$, to their assigned targets (m_j', or 0, or 1). Each p_t has exact type T, since $h_t > 0$ on $[0,1]$, and $p_t \to p'$. Hence all of $S(T')$ lies in $\overline{S(T)}$.

($\subseteq$) Let $p_n \in S(T)$, $p_n \to p$ nonconstant; then $B_n \to B$ and $g_n = B_n - p_n \to g = B - p \not\equiv 0$. Passing to a subsequence, we may assume each of the finitely many ordered contact positions of p_n converges in $[0,1]$. Choose disjoint small discs around the contacts z of g; around a contact of multiplicity μ_z, continuity of zeros for polynomials of bounded degree puts, for large n, exactly μ_z zeros of g_n (with multiplicity) in its disc: namely the contacts of p_n converging to z, plus zeros of the cofactor h_n. The h_n have no zeros in $[0,1]$, so their real zeros remain outside $[0,1]$: if z is interior, cofactor zeros reach it only as nonreal conjugate pairs—an even count—while at an endpoint any number may arrive one-sidedly. Assigning each interior contact of T to the contact of g it converges to (weakly monotone, as positions are ordered) produces exactly a certificate from T to the type of p. Finally, scaling $g \to t\,g$, $t \to 0^+$, shows the constants lie in every closure. ☐

Question 6.5. What remains is quantitative bookkeeping and one computation. (1) *Explicit chambers for $\mathcal{D}_5$.* In the chart $w = sx + \frac{u}{2}x^2 + \frac{r}{6}x^3 - x^4$, the tangency wall T_0 is exactly the surface

$$(s,u,r) \;=\; \bigl(m^2\rho,\ -2m^2-4m\rho,\ 12m+6\rho\bigr), \qquad m \in (0,1),\ \rho < 0,$$

coming from $w = -x\,(x-m)^2(x-\rho)$; eliminating the parameters, it lies on the hypersurface

$$\Delta_0(s,u,r) \;=\; 27\,N^2 + 2\,rND - 2\,uD^2 \;=\; 0, \qquad N = ru + 108\,s, \quad D = r^2 + 54\,u,$$

and the wall T_1 is its image under the chart reflection $(s,u,r) \mapsto (4-s-u-\frac{r}{2},\ u+r-12,\ 24-r)$. Assemble from these and from the planes $\{s=0\}$, $\{s+u+\frac{r}{2}=4\}$ the full chamber inequalities for $\mathcal{D}_5$, in the style of Theorem 5.6 (by the tangency bound of Theorem 6.2, only pairwise incidences occur). (2) *Measure.* Integrate the description of Theorem 5.6 to upgrade the degree-3 endpoint-maximum probability bound of [2] to an exact value.

Remark 6.6 (State of the bookkeeping). Both halves of Question 6.5 are now one step from the finish line. For (1), the critical-value identity reduces each chart of $\mathcal{D}_5$ to a *single* existential over an explicit algebraic function: writing w for the extremal quartic in the chart of leading coefficient ± 1, the data fails to force if and only if some interior local maximum $\sigma \in (0,1)$ of w satisfies the two quadratic inequalities

$$\begin{aligned} &(\text{chart } {+1})\text{:}\quad 18\sigma^2 + 2r\sigma + 3u \;<\; 0 \quad\text{and}\quad 18\sigma^2 + (2r+12)\sigma + (3u+r+6) \;<\; 0;\\ &(\text{chart } {-1})\text{:}\quad 18\sigma^2 - 2r\sigma - 3u \;>\; 0 \quad\text{and}\quad 18\sigma^2 + (12-2r)\sigma + (6-r-3u) \;>\; 0 \end{aligned}$$

(the first inequality of each pair says the maximum beats the level of the left endpoint, the second—obtained by factoring $w - w(\sigma)$ through its double root—that it beats the level of the right one). What remains is eliminating σ against the derivative cubic, a one-variable subresultant computation. For (2), the event of Theorem 5.6 is independent of c_0 and scale-invariant, hence a cone in (a_1,a_2,a_3)-space bounded by planes and a pair of mirror-image quadric cones—$4|a_3|\,a_1 \;\leq\; -(a_2^+)^2$ in the chart of negative leading coefficient, and its reflection $4a_3a_1 \geq (a_2 - 3a_3)(a_2 + a_3)$ in the positive chart; for i.i.d. standard Gaussian coefficients its probability reduces to one-dimensional integrals of the Gaussian distribution function between explicit boundaries, whose numerical evaluation (Gauss–Legendre quadrature, cross-checked by two independent Monte Carlo computations) gives

$$\mathrm{P}\big(\text{a random cubic attains its maximum over } [0,1] \text{ at an endpoint}\big) \;=\; 0.7802550079\ldots$$

($0.3140632\ldots$ from negative leading coefficient, $0.4661918\ldots$ from positive; we claim no digits beyond those displayed). The same reduction applies over any interval—for instance the probability is $0.7131644\ldots$ over $[-1,1]$ with the same model—so matching the precise normalization of [2] is a change of variables away.

The convex pieces of Theorem 2.6 can themselves be described completely. The cone $\mathcal{A}_n \cong \{p \in \mathbb{R}_{\le n-1}[x] : p \le p(0) \text{ on } [0,1]\}$ contains the line of constant polynomials, so it has no extreme rays as it stands; the pointed object is its quotient by the constants, realized concretely (via $q = p(0) - p$) as the cone

$$C_d \;=\; \big\{\, q \in \mathbb{R}_{\le d}[x] \;:\; q \ge 0 \text{ on } [0,1],\; q(0) = 0 \,\big\}, \qquad d = n-1.$$

Proposition 6.7. *The extreme rays of C_d are exactly the positive multiples of the polynomials*

$$q(x) \;=\; x^a\,(1-x)^b \prod_{i=1}^{\ell} (x - r_i)^{2m_i}$$

with integers $a \ge 1$, $b \ge 0$, $\ell \ge 0$, $m_i \ge 1$, distinct points $r_1, \ldots, r_\ell \in (0,1)$, and $a+b+2\sum_i m_i = d$: the polynomials spending their full degree on zeros in $[0,1]$.

Proof. Every $q \in C_d$, $q \not\equiv 0$, factors as $q = z\,p$, where $z(x) = x^a(1-x)^b \prod_i (x-r_i)^{2m_i}$ collects the zeros of q in $[0,1]$ (interior zeros have even multiplicity by nonnegativity, and $a \ge 1$ since $q(0) = 0$) and $p > 0$ on $[0,1]$.

If $\deg z < d$, then q is not extreme: the space $\mathbb{R}_{\le d - \deg z}[x]$ has dimension at least 2 and contains p, so choose g in it not proportional to p, and $\varepsilon > 0$ so small that $p \pm \varepsilon g > 0$ on $[0,1]$. Then $q_\pm = z\,(p \pm \varepsilon g)$ lie in C_d, are not multiples of q, and $q = \frac{1}{2}(q_+ + q_-)$.

If $\deg z = d$, then p is a positive constant and q is extreme: suppose $q = q_1 + q_2$ with $q_j \in C_d$. Then $0 \le q_j \le q$ on $[0,1]$, and a nonnegative polynomial bounded above by q near a zero of q in $[0,1]$ (one-sidedly at an endpoint) must vanish there to at least the order of q. Hence $z \mid q_j$, so $q_j = z\,p_j$ with $\deg p_j \le d - \deg z = 0$: each q_j is a nonnegative multiple of q.

This matches the classical description of the extreme rays of the cone of polynomials nonnegative on an interval (see, e.g., [5]); the constraint $q(0) = 0$ cuts out an exposed face, keeping exactly the rays with $a \ge 1$. Note that the double-contact cubics of Section 5 are the case $d = 3$, $(a, b, m_1) = (1, 0, 1)$—but x^d itself, with no interior contact, is extreme as well. ☐

Finally, we quantify how badly a function of the class with bounded data can overshoot its endpoint maximum—and here the answer is exact for every n. For $n \ge 3$ define

$$P_n(x) \;=\; \sum_{i=1}^{n-2} \frac{x^i - x^{n-1}}{i!},$$

the member of the class with data $(0, 1, 1, \ldots, 1;\, 0)$; it vanishes at both endpoints, and $P_3 = x - x^2$ is the counterexample that opened the paper. Let $E(n)$ be the supremum of

$$\max_{[0,1]} f \;-\; \max\{f(0),\, f(1)\}$$

over all $f \in C^{n-1}([0,1])$ with $f^{(n-1)}$ nondecreasing and $|f^{(i)}(0)| \le 1$ for $1 \le i \le n-2$; no constraint on $f(0)$ or $f(1)$ is imposed (none is needed).

Theorem 6.8 (The maximal overshoot, exactly)**.** *For every $n \geq 3$,*

$$E(n) \;=\; \max_{[0,1]} P_n,$$

attained by $f = P_n$ at the unique critical point $x_n \in (0,1)$ of P_n. Consequently, with $\sigma_k = \sum_{i=1}^{k} \frac{1}{i!}$:

(a) $E(3) = \frac{1}{4}$ *and* $E(4) = \dfrac{28 + 19\sqrt{19}}{243} = 0.45604\ldots$*;*
(b) $E(n)$ *is strictly increasing, via the identity* $P_{n+1} = P_n + \sigma_{n-1}\, x^{n-1}(1-x)$*;*
(c) $E(n) < \sigma_{n-2} < e - 1$ *for every n, and $E(n) \to e-1$ as $n \to \infty$.*

Proof. For f in the class with data d, Theorem 2.2 gives $f \leq v_d$ with the same endpoint values, so $E(n)$ is the supremum of $O(d) := \max_{[0,1]} v_d - \max\{c_0, \beta\}$ over $|c_i| \leq 1$ $(1 \leq i \leq n-2)$ and $c_0, \beta \in \mathbb{R}$.

Step 1: the interior derivatives saturate. For $1 \leq i \leq n-2$,

$$\frac{\partial v_d(x)}{\partial c_i} \;=\; \frac{x^i - x^{n-1}}{i!} \;\geq\; 0 \quad \text{on } [0,1],$$

while the benchmark $\max\{c_0, \beta\}$ does not involve c_i; so O is nondecreasing in each c_i and we may take $c_1 = \cdots = c_{n-2} = 1$.

Step 2: the endpoints balance. Subtracting c_0 shows O now depends only on $t := \beta - c_0$:

$$O(t) \;=\; \varphi(t) - t^{+}, \qquad \varphi(t) \;=\; \max_{x\in[0,1]} \Big[\sum_{i=1}^{n-2} \frac{x^i}{i!} + \big(t - \sigma_{n-2}\big) x^{n-1} \Big].$$

For $t_1 < t_2$, evaluating each maximum at the other's maximizer gives $\varphi(t_1) \leq \varphi(t_2) \leq \varphi(t_1) + (t_2 - t_1)$: thus φ is nondecreasing and $\varphi(t) - t$ is nonincreasing, whence $O(t) \leq O(0) = \varphi(0) = \max P_n$ for every $t \in \mathbb{R}$, with equality at $t = 0$—the data $(0, 1, \ldots, 1; 0)$ of P_n itself.

Uniqueness of the critical point. $P_n'(x) = \sum_{j=0}^{n-3} \frac{x^j}{j!} - (n-1)\,\sigma_{n-2}\, x^{n-2}$; dividing by x^{n-2} turns $P_n' = 0$ into $\sum_{j=0}^{n-3} x^{\,j-n+2}/j! = (n-1)\sigma_{n-2}$, whose left side is strictly decreasing on $(0,\infty)$. Since $P_n'(0) = 1 > 0 > P_n'(1)$, there is exactly one critical point $x_n \in (0,1)$, the maximum.

(a) For $n = 3$: $\max_{[0,1]}(x - x^2) = \frac{1}{4}$. For $n = 4$: $P_4 = x + \frac{x^2}{2} - \frac{3}{2}x^3$, and $P_4' = 0$ reads $9x^2 - 2x - 2 = 0$, so $x_4 = \frac{1+\sqrt{19}}{9}$; using $9x_4^2 = 2x_4 + 2$ to eliminate powers, $P_4(x_4) = \frac{19x_4+1}{27} = \frac{28+19\sqrt{19}}{243}$.

(b) $P_{n+1} - P_n = \frac{x^{n-1}}{(n-1)!} + \sigma_{n-2}\, x^{n-1} - \sigma_{n-1}\, x^n = \sigma_{n-1}\, x^{n-1}(1-x) \geq 0$, strictly positive at $x_n \in (0,1)$; so $E(n+1) \geq P_{n+1}(x_n) > P_n(x_n) = E(n)$.

(c) At the interior maximizer, $E(n) = P_n(x_n) < \sum_{i=1}^{n-2} x_n^i/i! \leq \sigma_{n-2} < e - 1$, the first inequality strict because $x_n^{n-1} > 0$. For fixed $x \in (0,1)$, $\sigma_{n-2}\, x^{n-1} \to 0$, so $P_n(x) \to e^x - 1$; hence $\liminf_n E(n) \geq e^x - 1$ for every $x < 1$, and $E(n) \to e - 1$. □

Remark 6.9. Adding the constraints $|f(0)| \leq 1$, $|f(1)| \leq 1$ of a full unit box changes nothing: the extremal P_n already satisfies them. The maximizer x_n is algebraic of degree at most $n-2$: expressible in radicals through $n = 6$ ($n = 5$: $40x^3 - 3x^2 - 6x - 6 = 0$), after which the characterization above remains; numerically $E(5) = 0.60958$, $E(6) = 0.72517$, $E(7) = 0.81521$. Thus the worst offender at every order is the canonical generalization of $x - x^2$: a truncated exponential pulled back to zero at the right endpoint.

Acknowledgments

This note revisits, and was prompted by rereading, the author's earlier note [7].

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