WHICH TWO-POINT HERMITE DATA FORCE THE MAXIMUM PRINCIPLE FOR $u^{(2m)} \geq 0$?

DAVID PAN

Abstract. Seventh in a series descending from [3]: we determine which balanced two-point Hermite data $u^{(k)}(a), u^{(k)}(b)$ $(0 \leq k \leq m-1)$ force the maximum principle for the class $u^{(2m)} \geq 0$—the one-dimensional clamped problem with independent data at the two ends, answering Problem 8.3 of [9]. The forcing set is a union of two convex cones exchanged by reflection, and each cone is, modulo the line of constant data, linearly isomorphic to the classical cone of polynomials of degree $\leq 2m-2$ nonnegative on an interval: semidefinite-representable for every m, and non-polyhedral already at $m=2$—with balanced data, curvature appears at the lowest nontrivial order, in contrast to the one-sided data of [4], where it waits until order four, and to the discrete case [6], where it never appears. For the clamped beam ($m=2$: $u'''' \geq 0$, prescribed endpoint heights and slopes) the criterion is a single inequality: with the height gap $\delta = u(b)-u(a) \geq 0$ and unit interval, the beam stays above its lower end if and only if

$$u'(a) \;\geq\; 0 \qquad \text{and} \qquad u'(b) \;\leq\; 3\,\delta + 2\sqrt{u'(a)\,\delta}\,,$$

so each chart is second-order-cone representable, with one curved quadric sheet and two flat faces: sharp constant 3, with a square-root enhancement from the slope at the low end. The two-point Hermite basis is one-signed on $[0,1]$ (a truncated binomial series), which yields an alternating criterion $(-1)^{m-k}\partial_\nu^k u \geq 0$ at each endpoint separately, at every order. The worst-case overshoot under unit data bounds is an explicit universal interpolant: it equals $1/4$ for $m=2$—the parabola $x(1-x)$ of [4] returns, mirrored, as the beam's worst sag $-x(1-x)$—and $11/32$ for $m=3$; it is bounded by $e-1$ for every m, its midpoint value tends to $\sqrt{e}-1$, and the midpoint is the exact maximizer for every $m \leq 24$ (verified in rational arithmetic), so up to a centrality conjecture the balanced ceiling is $\sqrt{e}-1$: exactly half the exponent of the one-sided ceiling $e-1$ of [4].

1. Introduction

Take an elastic beam clamped at both ends—heights and slopes prescribed—and push it upward everywhere ($u'''' \geq 0$, in Euler–Bernoulli form). Can it still sag below the lower of its two supports? The answer, it turns out, is a single inequality in the four boundary numbers: sagging is impossible exactly when the slope at the low end points into the span and the slope at the high end obeys

$$u'(\text{high end}) \;\leq\; 3\,(\text{height gap}) \;+\; 2\sqrt{u'(\text{low end})\,(\text{height gap})}\,.$$

This paper proves that criterion (Theorem 5.1), places it in a complete forcing theory for the classes $u^{(2m)} \geq 0$ with balanced two-point Hermite data at every order m, and computes the exact worst-case overshoot, whose large-m ceiling is—conjecturally, with centrality verified through $m=24$—the constant $\sqrt{e}-1$.

Date: August 9, 2026.

2020 *Mathematics Subject Classification.* Primary 26A51, 26D10; Secondary 34B27, 41A05.

Key words and phrases. maximum principle, Hermite interpolation, clamped beam, nonnegative polynomials, two-point boundary value problems, overshoot.

The question continues the program of [4, 5, 6, 7, 8, 9]: for a class defined by a differential inequality and prescribed data, which data *force* the maximum principle? The immediate predecessor [9] developed the theory of Dirichlet (clamped) data for $\Delta^m u \geq 0$ and showed that in one dimension it concerns the balanced two-point Hermite pattern—m conditions at each end—which none of the earlier papers covers: [4, 5, 6] prescribe $n-1$ conditions at one end and one at the other, and [8] prescribes even-order derivatives (the Lidstone pattern). But [9] solved only the symmetric slice (constant data on the ball, which in $d = 1$ means reflection-even data), and its Problem 8.3 asked for the full asymmetric cone. Here it is. Three structural points:

- **Two charts, one classical cone** (Theorem 3.1). The forcing set splits by which endpoint carries the benchmark; each chart, after dividing the barrier by its forced endpoint root and passing to the quotient by the constants line, is linearly isomorphic to the cone of polynomials of degree $\leq 2m-2$ nonnegative on $[0, 1]$—the same classical object that appeared for constant ball data in [9], but with twice the degree. Consequence: the cone is semidefinite-representable for every m, and non-polyhedral *already for* $m = 2$. Balanced data buys curvature at the first nontrivial order: the one-sided cones of [4] are polyhedral through order three, the discrete cones of [6] are polyhedral forever, and the balanced cone never is.
- **The Hermite basis is one-signed** (Lemma 4.1). The (m, m) Hermite cardinal functions are truncated binomial series, $H_{k0} = \frac{x^k}{k!}(1-x)^m \sum_{j \leq m-1-k} \binom{m-1+j}{j} x^j$, hence positive. This gives in three lines an alternating criterion with the same sign pattern $(-1)^{m-k}\partial_\nu^k u \geq 0$ as [8, Theorem 3.1] and [9, Theorem 4.7]—but now imposed pointwise at each endpoint separately, with no symmetry between the ends.
- **The overshoot ceiling halves its exponent** (Theorem 7.1, Theorem 7.2). The worst overshoot under unit data bounds is the maximum of one universal interpolant S_m. Exactly: $E_2 = \frac{1}{4}$, attained by the sag $-x(1-x)$—the parabola that opened this series in [4], mirrored—and $E_3 = \frac{11}{32}$. Asymptotically: $E_m < e - 1$ always, $S_m(\frac{1}{2}) \to \sqrt{e} - 1$, and the midpoint is the exact maximizer for every $m \leq 24$ (verified in rational arithmetic), so—modulo the centrality conjecture of Problem 8.2—the balanced ceiling is $\sqrt{e} - 1$ against the one-sided $e - 1$ of [4]: splitting the data evenly halves the exponent. We conjecture the pattern $e^{\max(p,n-p)/n} - 1$ for a general $(p, n-p)$ split (Problem 8.3).

Section 2 sets up the reduction (from [9]); Section 3 proves the chart structure; Section 4 the basis positivity and the alternating criterion; Section 5 the complete $m = 2$ answer; Section 6 the order-three reduction; Section 7 the overshoot; Section 8 collects problems. All exact claims are verified in rational arithmetic; the script accompanies the source.

2. Setting and the reduction

Work on $[0, 1]$; general intervals follow by the affine substitution recorded in Remark 5.3. Fix $m \geq 1$ and data $D = (\alpha; \beta) = (\alpha_0, \ldots, \alpha_{m-1}; \beta_0, \ldots, \beta_{m-1}) \in \mathbb{R}^{2m}$, and let

$$\mathcal{K}_m(D) \;=\; \big\{u \in C^{2m}([0,1]) \;:\; u^{(2m)} \geq 0,\ \ u^{(k)}(0) = \alpha_k,\ u^{(k)}(1) = \beta_k\ (0 \leq k \leq m-1)\big\}.$$

D is *max-forcing* if every member satisfies $u \leq \max\{\alpha_0, \beta_0\}$, *min-forcing* if every member satisfies $u \geq \min\{\alpha_0, \beta_0\}$. Let v_D be the (m, m) two-point Hermite interpolant: the unique polynomial of degree $\leq 2m-1$ with the data D.

This is the one-dimensional clamped problem of [9], whose dichotomy applies with the positivity input supplied by classical disconjugacy (the (m, m) two-point Green function of $u^{(2m)}$ has sign $(-1)^m$; see [1, Ch. III, §5] and [9, §7]):

Theorem 2.1 (Reduction; [9, Lemma 2.3, Theorem 3.1, and §7])**.** *If m is even, v_D is the pointwise-least element of $\mathcal{K}_m(D)$, and D is min-forcing iff $v_D \geq \min\{\alpha_0, \beta_0\}$ on $[0, 1]$; no*

D is max-forcing. If m is odd, v_D is the pointwise-greatest element, D is max-forcing iff $v_D \le \max\{\alpha_0, \beta_0\}$, and no D is min-forcing.

Write $\mathcal{D}_m \subset \mathbb{R}^{2m}$ for the set of forcing data (min-forcing for m even, max-forcing for m odd). Two symmetries act. Translation: adding a constant to u shifts (α_0, β_0) together and preserves forcing, so $\mathcal{D}_m$ contains the line of constants. Reflection: $\rho u(x) := u(1-x)$ maps the class to itself with data

$$\rho(\alpha;\beta) \;=\; \big(((-1)^k\beta_k)_k\,;\,((-1)^k\alpha_k)_k\big),$$

and $\mathcal{D}_m$ is ρ-invariant.

3. Two charts, one classical cone

Theorem 3.1 (Structure)**.** *Fix $m \ge 1$ and set $\delta := \beta_0 - \alpha_0$. Define the chart*

$$\mathcal{A}_m \;:=\; \begin{cases} \{D : v_D \ge \alpha_0 \text{ on } [0,1]\} & (m \text{ even: benchmark at the left end}),\\ \{D : v_D \le \beta_0 \text{ on } [0,1]\} & (m \text{ odd: benchmark at the right end}). \end{cases}$$

Then:

(i) *$\mathcal{D}_m = \mathcal{A}_m \cup \rho\,\mathcal{A}_m$, and $\mathcal{A}_m \subseteq \{\delta \ge 0\}$ automatically.*

(ii) *The map $D \mapsto \chi_D := (v_D - \alpha_0)/x$ (for m even; $\chi_D := (v_D - \beta_0)/(x-1)$ for m odd) is linear and surjective onto the polynomials of degree $\le 2m-2$, with kernel the line of constants, and*

$$\mathcal{A}_m \;=\; \{D : \chi_D \ge 0 \text{ on } [0,1]\}.$$

Thus $\mathcal{A}_m$, modulo its lineality line, is linearly isomorphic to the classical cone $\mathcal{P}_{2m-2}$ of polynomials of degree $\le 2m-2$ nonnegative on $[0,1]$.

(iii) *$\mathcal{A}_m$ is a closed convex cone, semidefinite-representable by the Markov–Lukács theorem* [10, §1.21]*; its boundary consists of the data whose χ_D has a zero in $[0,1]$ (the barrier grazes the benchmark level). The classical extreme structure of* [2] *lives on the quotient by the lineality line of constants ($\mathcal{A}_m$ itself contains that line, so it has no extreme rays in the literal sense).*

(iv) *For every $m \ge 2$, $\mathcal{A}_m$ is not polyhedral, and neither is $\mathcal{D}_m$; and $\mathcal{D}_m$ is not convex.*

Proof. Take m even (the odd case mirrors, exchanging the roles of the two ends). (i) If $\delta \ge 0$ the benchmark is $\min\{\alpha_0, \beta_0\} = \alpha_0$ and Theorem 2.1 says forcing $\iff$ $v_D \ge \alpha_0$; if $\delta \le 0$, apply ρ. Membership $v_D \ge \alpha_0$ evaluated at $x = 1$ gives $\beta_0 \ge \alpha_0$, so the chart condition implies $\delta \ge 0$ by itself.

(ii) $\psi := v_D - \alpha_0$ vanishes at 0, so $\chi_D = \psi/x$ is a polynomial of degree $\le 2m-2$ depending linearly on D; and $\psi \ge 0$ on $[0,1]$ iff $\chi_D \ge 0$ on $(0,1]$, iff on $[0,1]$ by continuity. The kernel of $D \mapsto \chi_D$ is $\{v_D \equiv \alpha_0\}$, the constants line; the domain has dimension $2m$ and the target $2m-1$, so the map is onto.

(iii) $\mathcal{A}_m$ is the preimage of $\mathcal{P}_{2m-2}$ under the continuous linear map $D \mapsto \chi_D$, hence closed and convex, and it inherits the Markov–Lukács semidefinite representation (compose the representation with the linear map); the induced map on the quotient by the constants line is a linear isomorphism onto $\mathcal{P}_{2m-2}$, which carries the boundary and extreme structure. A point of $\mathcal{P}_{2m-2}$ is interior iff the polynomial is strictly positive on $[0,1]$.

(iv) The section of $\mathcal{P}_{2m-2}$ ($2m-2 \ge 2$) by the subspace of quadratics is the cone of quadratics nonnegative on $[0,1]$, which has the continuum of extreme rays $(x-s)^2$, $s \in [0,1]$ [2]; polyhedral cones have polyhedral sections with finitely many extreme rays, so $\mathcal{A}_m$ is not polyhedral. Since $\mathcal{A}_m = \mathcal{D}_m \cap \{\delta \ge 0\}$—one inclusion is (i), the other is that a forcing D with $\delta \ge 0$ satisfies the chart condition—polyhedrality of $\mathcal{D}_m$ would force it for $\mathcal{A}_m$.

Nonconvexity. First record two necessary conditions: on $\mathcal{A}_m$ (m even), $\chi_D(0) = (v_D - \alpha_0)'(0) = \alpha_1 \geq 0$; on the reflected chart, $-\beta_1 \geq 0$. Now note $H_{10} \leq H_{00}$ on $[0,1]$, because $xS_{m,1} \leq S_{m,0}$ termwise in the binomial sums of Lemma 4.1. For m even, let a be the data with all left data zero, $\beta_0 = \beta_1 = 1$, other right data zero: its barrier is $H_{01} + H_{11} = H_{00}(1-x) - H_{10}(1-x) \geq 0$, so a forces, as does ρa; but their midpoint has $\delta = 0$, $\alpha_1 = -\frac{1}{2}$ and $\beta_1 = \frac{1}{2}$, violating the necessary conditions of both charts at a benchmark tie. For m odd, the data $\alpha_0 = -1$, $\alpha_1 = 1$, all else zero (barrier $-H_{00} + H_{10} \leq 0$) plays the same role against its reflection. $\square$

Remark 3.2. Compare the geometry across the series: the one-sided cones $\mathcal{D}_n$ of [4] are polyhedral for $n \leq 3$ and curved from $n = 4$; the discrete cones of [6] are polyhedral for every order; the constant-data ball cones of [9] are $\mathcal{P}_{m-2}$, polyhedral through $m = 3$. Balanced two-point data is the extreme case: $\mathcal{P}_{2m-2}$ is curved as soon as $m = 2$. In terms of the differential order $n = 2m$, both [4] and the present paper first curve at $n = 4$—the two data patterns of order four, $(3,1)$ and $(2,2)$, both produce non-polyhedral cones, one barely, one structurally.

4. The Hermite basis is one-signed; the alternating criterion

Let H_{k0} and H_{k1} ($0 \leq k \leq m-1$) be the (m,m) Hermite cardinal polynomials on $[0,1]$: $H_{k0}^{(j)}(0) = \delta_{jk}$, $H_{k0}^{(j)}(1) = 0$ and symmetrically for H_{k1}, for $0 \leq j \leq m-1$.

Lemma 4.1 (Binomial form and positivity)**.** *For $0 \leq k \leq m-1$,*

$$H_{k0}(x) \;=\; \frac{x^k}{k!}\,(1-x)^m \sum_{j=0}^{m-1-k} \binom{m-1+j}{j} x^j, \qquad H_{k1}(x) \;=\; (-1)^k H_{k0}(1-x).$$

In particular $H_{k0} \geq 0$ and $(-1)^k H_{k1} \geq 0$ on $[0,1]$, every cardinal is one-signed, and $H_{00} + H_{01} \equiv 1$.

Proof. The displayed function has degree $k + m + (m-1-k) = 2m-1$ and vanishes to order m at $x = 1$. At $x = 0$: the sum $S(x)$ is the truncation of the binomial series $(1-x)^{-m} = \sum_{j\geq 0} \binom{m-1+j}{j} x^j$, so

$$(1-x)^m S(x) \;=\; 1 - (1-x)^m \sum_{j \geq m-k} \binom{m-1+j}{j} x^j \;=\; 1 + O(x^{m-k}),$$

whence $H_{k0}(x) = \frac{x^k}{k!} + O(x^m)$: the derivatives of orders $0, \ldots, m-1$ at 0 are exactly δ_{jk}. Uniqueness of Hermite interpolation identifies the cardinal. The reflection identity is the chain rule plus uniqueness; positivity is visible in the formula; and $H_{00} + H_{01}$ interpolates the constant 1. $\square$

In the cardinal basis, the chart barrier of Theorem 3.1 splits into one-signed pieces: for m even and $\delta \geq 0$,

$$v_D - \alpha_0 \;=\; \sum_{k=1}^{m-1} \alpha_k H_{k0} \;+\; \delta\, H_{01} \;+\; \sum_{k=1}^{m-1} \beta_k H_{k1}, \tag{1}$$

using $H_{00} + H_{01} = 1$. Writing ∂_ν for the outward derivative at an endpoint ($\partial_\nu^k u(0) = (-1)^k \alpha_k$, $\partial_\nu^k u(1) = \beta_k$), termwise signs give:

Theorem 4.2 (Alternating criterion)**.** *If*

$$(-1)^{m-k}\, \partial_\nu^k u \;\geq\; 0 \quad \textit{at both endpoints}, \qquad 1 \leq k \leq m-1,$$

then D is forcing (min- for m even, max- for m odd), for arbitrary endpoint values α_0, β_0. The sign pattern is that of [8, Theorem 3.1] *and* [9, Theorem 4.7]*, now imposed at each endpoint separately, with independent data.*

Proof. Let m be even and $\delta \geq 0$ (else reflect; the hypothesis is ρ-invariant). The hypothesis reads $\alpha_k \geq 0$ and $(-1)^k \beta_k \geq 0$ for $1 \leq k \leq m-1$. In (1), each term is then nonnegative by Lemma 4.1, so $v_D \geq \alpha_0 = \min\{\alpha_0, \beta_0\}$, and Theorem 2.1 concludes. For m odd all signs flip. □

5. The clamped beam: a complete answer

Theorem 5.1 (The beam criterion)**.** *Let $m = 2$: $u'''' \geq 0$ on $[0,1]$ with prescribed heights α_0, β_0 and slopes α_1, β_1. Suppose $\alpha_0 \leq \beta_0$ and set $\delta = \beta_0 - \alpha_0$. Then D is min-forcing if and only if*

$$\alpha_1 \;\geq\; 0 \qquad \text{and} \qquad \beta_1 \;\leq\; 3\,\delta + 2\sqrt{\alpha_1 \delta}\,,$$

equivalently, in radical-free form, $\alpha_1 \geq 0$ and $\big[\, \beta_1 \leq 3\delta$ or $(\beta_1 - 3\delta)^2 \leq 4\,\alpha_1 \delta \,\big]$. The case $\alpha_0 \geq \beta_0$ is the reflection $(\alpha_1, \beta_1, \delta) \mapsto (-\beta_1, -\alpha_1, -\delta)$.

Proof. By Theorem 3.1, min-forcing means $\chi := (v_D - \alpha_0)/x \geq 0$ on $[0,1]$, where by (1) and the $m = 2$ cardinals ($H_{10} = x(1-x)^2$, $H_{01} = x^2(3-2x)$, $H_{11} = x^2(x-1)$),

$$\chi(x) \;=\; \alpha_1 (1-x)^2 + \delta\, x(3-2x) + \beta_1\, x(x-1).$$

At the ends, $\chi(0) = \alpha_1$ and $\chi(1) = \delta$; so $\alpha_1 \geq 0$ is necessary, and $\delta \geq 0$ holds in the chart. For $x \in (0,1)$, dividing $\chi(x) \geq 0$ by $x(1-x) > 0$ and rearranging,

$$\chi \geq 0 \;\text{ on } (0,1) \iff \beta_1 \;\leq\; \inf_{0<x<1} \Big[\alpha_1 \frac{1-x}{x} + \delta\, \frac{3-2x}{1-x} \Big] \;=\; \inf_{0<x<1} \Big[\frac{\alpha_1}{x} + \frac{\delta}{1-x} \Big] - \alpha_1 + 2\delta,$$

using $\frac{3-2x}{1-x} = 2 + \frac{1}{1-x}$. By Cauchy–Schwarz,

$$\frac{\alpha_1}{x} + \frac{\delta}{1-x} \;=\; \Big(\frac{\alpha_1}{x} + \frac{\delta}{1-x} \Big) \big(x + (1-x) \big) \;\geq\; \big(\sqrt{\alpha_1} + \sqrt{\delta} \big)^2,$$

with equality at $x = \sqrt{\alpha_1}/(\sqrt{\alpha_1} + \sqrt{\delta}) \in (0,1)$ when $\alpha_1, \delta > 0$. If exactly one of α_1, δ vanishes, the infimum $(\sqrt{\alpha_1} + \sqrt{\delta})^2$ is still correct but is only approached, at the endpoint $x \downarrow 0$ (if $\alpha_1 = 0$) or $x \uparrow 1$ (if $\delta = 0$); if both vanish it is 0. So the infimum equals $(\sqrt{\alpha_1} + \sqrt{\delta})^2 - \alpha_1 + 2\delta = 3\delta + 2\sqrt{\alpha_1 \delta}$. Continuity extends the equivalence from $(0,1)$ to $[0,1]$. Squaring where $\beta_1 - 3\delta > 0$ gives the radical-free form. □

Remark 5.2 (Sharp constant 3, and the geometry). If the low end is clamped flat ($\alpha_1 = 0$), the criterion is $\beta_1 \leq 3\delta$: the slope at the high end may not exceed *three times* the height gap. The extremal is $v = \alpha_0 + \delta x^3$, grazing the low support to third order. A positive low-end slope buys the enhancement $2\sqrt{\alpha_1 \delta}$: the sharp constant is 3, and the correction is exactly the Cauchy–Schwarz defect. In the coordinates $(x, y, z) = (\alpha_1,\, \beta_1 - 3\delta,\, \delta)$ the chart is $\{x, z \geq 0,\; y \leq 2\sqrt{xz}\}$: the *one-sided* shadow of the rotated Lorentz cone $\{y^2 \leq 4xz\}$, with one curved quadric sheet and two flat faces (the data $(1, -10, 1)$ lies in the chart but not in the Lorentz cone). It is second-order-cone representable by a one-variable lift: $\exists\, t \geq 0$ with $y \leq t$ and $t^2 \leq 4xz$—the series' first forcing cone whose curved boundary is a single quadric sheet. The boundary rays with interior contact are

$$(\alpha_1, \beta_1, \delta) \;=\; \big(s^2,\; (1-s)(3-s),\; (1-s)^2 \big), \qquad s \in [0,1]\,:$$

here $v - \alpha_0 = x\,(x-s)^2$, a beam grazing its lower support height at the interior point s. At $\delta = 0$ the criterion degenerates to $\alpha_1 \geq 0 \geq \beta_1$—both slopes point into the span—recovering the symmetric slice [9, Theorem 4.5(iii)] at both ends independently.

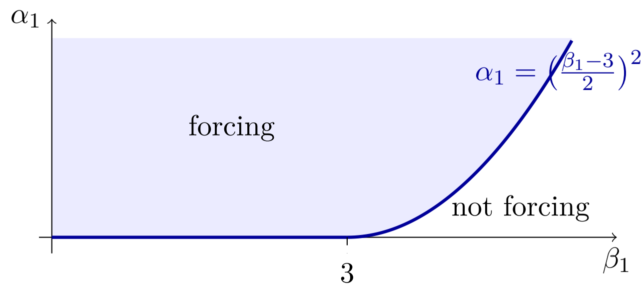

FIGURE 1. The slice $\delta = 1$ of the beam chart (Theorem 5.1): the forcing region is the epigraph of a flat segment followed by a parabola. Points on the curved boundary are beams $v = \alpha_0 + x(x-s)^2$ grazing the lower support at an interior point.

Remark 5.3 (General intervals). On $[a, b]$ with $L = b - a$, the substitution $x \mapsto a + Lx$ scales $u^{(k)}$ by L^k; the beam criterion becomes $u'(a) \geq 0$ and $L\,u'(b) \leq 3\delta + 2\sqrt{L\,u'(a)\,\delta}$ with $\delta = u(b) - u(a) \geq 0$.

6. Order three: the quartic chart

For $m = 3$ ($u^{(6)} \geq 0$, max-forcing), the chart with $\beta_0 \geq \alpha_0$ reduces by Theorem 3.1 to the nonnegativity on $[0, 1]$ of the quartic $\chi = (v_D - \beta_0)/(x-1)$, with anchors $\chi(0) = \delta$ and $\chi(1) = \beta_1$. Two immediate consequences: $\beta_1 \geq 0$ is *necessary* (the slope at the benchmark end must point outward—if $\beta_1 < 0$ the barrier exceeds β_0 just inside the right end), and the alternating orthant of Theorem 4.2 is sufficient. Membership is decidable for any given data by classical quartic analysis (or two semidefinite blocks via Markov–Lukács); what is missing is the explicit chamber description in the six data coordinates—the balanced counterpart of the order-four answer of [4, §5]—which we leave as Problem 8.1. The cardinals, for use below: $H_{10} = x(1-x)^3(1+3x)$, $H_{20} = \frac{1}{2}x^2(1-x)^3$.

7. The overshoot

How far can a member exceed the benchmark when the non-value data are bounded by one? Define, for m odd,

$$E_m \;=\; \sup\Big\{\max_{[0,1]} u - \max\{\alpha_0, \beta_0\} \;:\; u \in \mathcal{K}_m(D),\; |\alpha_k|, |\beta_k| \leq 1\ (1 \leq k \leq m-1)\Big\},$$

and for m even the mirror quantity $\sup(\min\{\alpha_0, \beta_0\} - \min u)$.

Theorem 7.1 (Overshoot)**.** *For every $m \geq 2$,*

$$E_m \;=\; \max_{[0,1]} S_m, \qquad S_m \;:=\; \sum_{k=1}^{m-1}\big[H_{k0}(x) + H_{k0}(1-x)\big],$$

and $(-1)^{m+1}S_m$ is itself the barrier of the data $\alpha_0 = \beta_0 = 0$, $\alpha_k = (-1)^{m+1}$, $\beta_k = (-1)^{m+k+1}$: one universal extremal for each order, realizing the overshoot $+S_m$ for m odd and the sag $-S_m$ for m even. Exactly:

$$S_2 = s, \qquad S_3 = s + \tfrac{3}{2}s^2 \qquad (s := x(1-x)),$$

so $E_2 = \frac{1}{4}$, attained by the sag $-x(1-x)$, and $E_3 = \frac{11}{32}$; further $E_4 = \frac{151}{384}$, $E_5 = \frac{2609}{6144}$, $E_6 = \frac{54901}{122880}$, all attained at the midpoint (each value certified in exact arithmetic via the expansion of S_m in s, whose coefficients are nonnegative for every $m \leq 24$).

Proof. Let m be odd (even mirrors). By Theorem 2.1 and (1)'s analogue at the max end, $u \le v_D \le \max\{\alpha_0, \beta_0\} + \sum_{k\ge 1}\big[|H_{k0}| + |H_{k1}|\big]$ pointwise, using $\alpha_0 H_{00} + \beta_0 H_{01} \le \max\{\alpha_0, \beta_0\}$ (positivity and partition of unity, Lemma 4.1). By the reflection identity $|H_{k1}(x)| = H_{k0}(1-x)$, the bound is S_m, and it is attained: the data $\alpha_k = 1$, $\beta_k = (-1)^k$ produce the barrier $\sum[H_{k0} + (-1)^k H_{k1}] = S_m$ identically—no pointwise sign choice is needed—which is a member of its own class with benchmark 0. For m even the mirrored quantity is realized by the negated data $\alpha_k = -1$, $\beta_k = (-1)^{k+1}$, whose barrier is $-S_m$: the sag below the benchmark 0 is again $\max S_m$. The displayed s-forms follow from the cardinals: for $m = 2$, $H_{10}(x) + H_{10}(1-x) = x(1-x)^2 + x^2(1-x) = s$; for $m = 3$, $H_{10}(x) + H_{10}(1-x) = x(1-x)(1+x-x^2) = s(1+s)$ and $H_{20}(x) + H_{20}(1-x) = \frac{1}{2}x^2(1-x)^2 = \frac{1}{2}s^2$. Since s-expansions with nonnegative coefficients are maximized where s is, namely the midpoint $s = \frac{1}{4}$, the values follow. □

Theorem 7.2 (The ceiling)**.** *For every m, $E_m < e-1$. Moreover*

$$S_m\big(\tfrac{1}{2}\big) \;\longrightarrow\; \sqrt{e}-1 \qquad (m\to\infty),$$

with error $O(m^{-1/2})$; hence $\liminf_m E_m \ge \sqrt{e}-1$, and if the midpoint remains the maximizer for all m (as verified through $m = 24$), then $E_m \to \sqrt{e}-1$.

Proof. Upper bound. Since the truncated binomial sum for H_{k0} is termwise dominated by the one for H_{00},

$$H_{k0}(x) \le \frac{x^k}{k!}H_{00}(x), \qquad H_{k0}(1-x) \le \frac{(1-x)^k}{k!}H_{00}(1-x) = \frac{(1-x)^k}{k!}H_{01}(x),$$

so by the partition of unity, $H_{k0}(x) + H_{k0}(1-x) \le \max(x, 1-x)^k/k!$, and

$$S_m(x) \;\le\; e^{\max(x,1-x)} - 1 \;<\; e-1 \qquad (0<x<1),$$

while $S_m = 0$ at the endpoints; a continuous function on $[0,1]$ that is everywhere strictly below $e-1$ has maximum below $e-1$.

Midpoint limit. The engine is the identity

$$\sum_{j=0}^{m-1}\binom{m-1+j}{j}2^{-j} \;=\; 2^{m-1}, \tag{2}$$

seen probabilistically: in fair coin flips, the event "m heads occur before m tails" has probability $\frac{1}{2}$ by symmetry, and decomposing it by the number $j \le m-1$ of tails at the moment of the m-th head gives $\sum_j \binom{m-1+j}{j}2^{-(m+j)} = \frac{1}{2}$. Hence

$$2H_{k0}\big(\tfrac{1}{2}\big) \;=\; \frac{2^{-k}}{k!}\,2^{1-m}\sum_{j=0}^{m-1-k}\binom{m-1+j}{j}2^{-j} \;=\; \frac{2^{-k}}{k!}\Big(1 - 2^{1-m}M_{m,k}\Big),$$

where $M_{m,k} = \sum_{j=m-k}^{m-1}\binom{m-1+j}{j}2^{-j}$ collects the k missing terms. Each is at most the largest, $\binom{2m-2}{m-1}2^{-(m-1)}$, and $\binom{2n}{n} \le 4^n/\sqrt{\pi n}$ gives $2^{1-m}M_{m,k} \le k/\sqrt{\pi(m-1)}$. Therefore

$$S_m\big(\tfrac{1}{2}\big) \;=\; \sum_{k=1}^{m-1}\frac{2^{-k}}{k!} \;-\; O\Big(\frac{1}{\sqrt{m}}\sum_{k\ge 1}\frac{k\,2^{-k}}{k!}\Big) \;\longrightarrow\; \sum_{k=1}^{\infty}\frac{2^{-k}}{k!} \;=\; e^{1/2}-1. \qquad \square$$

Remark 7.3. Three echoes. First, $E_2 = \frac{1}{4}$ with extremal $-x(1-x)$ is numerically *and* functionally the constant $E(3) = \frac{1}{4}$ of [4]: the same parabola, up to the parity-mandated sign, is the worst case for the one-sided order-three problem and the balanced order-four one. Second, the one-sided overshoots of [4] satisfy $E(n) \uparrow e-1$ unconditionally, while the balanced ceiling is $\sqrt{e}-1$ *modulo centrality* (Theorem 7.2 proves the strict bound $E_m < e-1$ and $\liminf E_m \ge \sqrt{e}-1$):

distributing the data evenly over the two ends should exactly halve the exponent. Third, for m odd the same universal data realize an upward overshoot: overshoot and sag alternate with the parity, as they must.

8. Problems

Problem 8.1. Order three, explicitly. Describe the chambers of the quartic chart of Section 6 in the six data coordinates: the balanced counterpart of the complete order-four description of [4, §5]. The reflection now acts *between* the two charts rather than within one, so no half-of-the-work symmetry is available; a two-chart analysis in the style of [5, Problem 6.2] is needed.

Problem 8.2. Centrality. Show that the expansion of S_m in $s = x(1-x)$ has nonnegative coefficients for every m (verified exactly for $m \le 24$), so that $E_m = S_m(\frac{1}{2}) \uparrow \sqrt{e} - 1$; and determine the sharp rate, numerically $\asymp m^{-1/2}$.

Problem 8.3. General splits. For $u^{(n)} \ge 0$ with two-point Hermite data split $(p, n-p)$, the $(p, n-p)$ Green function has sign $(-1)^{n-p}$ (classical disconjugacy [1]), so every split has a one-signed barrier and a forcing theory: [4] is the split $(n-1, 1)$ and this paper is (m, m). Unify: chart structure, alternating criteria, and above all the overshoot ceiling. The proven one-sided ceiling $e-1$ (split fraction $\to 1$, [4]) and the conjectural balanced ceiling $\sqrt{e}-1$ (at $\frac{1}{2}$, Theorem 7.2 and Problem 8.2) suggest $E^{(p,n-p)} \to e^{\max(p,n-p)/n} - 1$ along fixed-ratio sequences.

Problem 8.4. Disconjugate version. Replace $u^{(2m)}$ by a disconjugate operator in Pólya form with quasi-derivative data at both ends, as [5] did for the one-sided problem. The sign of the (m, m) Green function persists under disconjugacy [1], so Theorem 2.1 should survive verbatim; which parts of the explicit $m = 2$ answer do? In particular, is there always a "sharp constant 3" — i.e., does $\sup \beta_1/\delta$ over the flat-low-end chart equal a generalized trapezoid-type weight as in [5]?

Acknowledgments

Seventh in a series descending from [3], after [4, 5, 6, 7, 8, 9]; it answers Problem 8.3 of [9]. All exact constants were verified in rational arithmetic.

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