

FORCING SETS FOR MAXIMUM PRINCIPLES OF DISCONJUGATE OPERATORS

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ABSTRACT. The companion paper [6] determined which endpoint data $f(a), \dots, f^{(n-2)}(a), f(b)$ force the maximum principle $f \leq \max\{f(a), f(b)\}$ for functions with $f^{(n)} \geq 0$. We show the core of the theory never depended on polynomials. For any disconjugate operator L in Pólya form that annihilates constants, the class of functions with $Lf \geq 0$, prescribed quasi-derivative data at a , and prescribed value at b has a pointwise-greatest element—the interpolant from the kernel of L —so the forcing problem again reduces to the maximum principle for a single explicit function, and the forcing set is again a union of two convex cones. The proof is a sign-chain argument through the Pólya quasi-derivatives which simultaneously yields the descending-data criterion, and is simpler than the original arguments even for $L = D^n$. For operators of order three we determine the forcing set completely: the sharp criterion is $f'(a) \leq \alpha_L^{-1}(f(b) - f(a))^+$, where α_L is the left weight of the unique generalized trapezoidal rule exact on $\ker L$; for $L = D^3$ this is the constant 2 of [6]. The boundary of the forcing cone is characterized as in the polynomial case (for $n \geq 3$), and the framework applies verbatim to radial polyharmonic functions on annuli.

1. INTRODUCTION

The note [5] proved that a function with $f^{(n)} \geq 0$ on $[a, b]$ satisfies the *maximum principle* $f \leq \max\{f(a), f(b)\}$ under suitable endpoint conditions, and the companion paper [6] determined exactly which data

$$(f(a), f'(a), \dots, f^{(n-2)}(a); f(b))$$

force it: the class of admissible functions has a pointwise-greatest element, an explicit polynomial of degree at most $n-1$, so forcing is decided by the maximum principle for that single polynomial; the resulting forcing set is a union of two convex cones, is described by the sharp inequality $f'(a)(b-a) \leq 2 \max\{f(b) - f(a), 0\}$ when $n = 3$, and carries a completely analyzed boundary stratification.

Inspecting those proofs, the central ones—the greatest-element reduction, the descending-data criterion, the two-cone decomposition, and the coarse boundary characterization—never use polynomials as such. (The finer contact stratification of [6] does lean on polynomial factorization; see Remark 4.3 below.) They use two facts: that solutions of $f^{(n)} = 0$ interpolate any n Hermite conditions uniquely, and that the relevant interpolation error has a sign. Both are the defining properties of *disconjugacy*. This paper redoes the theory at that level of generality.

Throughout, fix $n \geq 2$, a compact interval $[a, b]$, and weights

$$w_1, \dots, w_{n-1} \in C^n[a, b], \quad w_i > 0 \text{ on } [a, b].$$

Define the *quasi-derivatives*

$$L_0 f = f, \quad L_i f = \frac{(L_{i-1} f)'}{w_i} \quad (1 \leq i \leq n-1), \quad Lf := (L_{n-1} f)'.$$

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Thus L is an n -th order linear operator with $L1 = 0$; by Pólya's factorization theorem [7, 1], every disconjugate operator on a compact interval that annihilates constants arises this way, up to multiplication by a positive function (which does not affect the sign of Lf). For $w_i \equiv 1$ we recover $L = D^n$. The kernel

$$\mathcal{N} = \{u : Lu = 0\} = \{u : L_{n-1}u \equiv \text{const}\}$$

is spanned by the *canonical basis* $u_0 = 1$ and

$$u_j(x) = \int_a^x w_1(t_1) \int_a^{t_1} w_2(t_2) \cdots \int_a^{t_{j-1}} w_j(t_j) dt_j \cdots dt_1, \quad 1 \leq j \leq n-1,$$

which satisfies $L_i u_j(a) = \delta_{ij}$ and $u_j > 0$ on $(a, b]$.

Definition 1.1. A function f is *admissible* if $L_0 f, \dots, L_{n-1} f$ are defined and continuous on $[a, b]$. For a data vector $d = (c_0, \dots, c_{n-2}; \beta) \in \mathbb{R}^n$ put

$$\mathcal{K}_L(d) = \{f \text{ admissible} : L_{n-1}f \text{ nondecreasing, } L_i f(a) = c_i \ (0 \leq i \leq n-2), \ f(b) = \beta\},$$

a class containing every C^n function with $Lf \geq 0$ and the given data. The data d is *forcing* if every $f \in \mathcal{K}_L(d)$ satisfies the maximum principle on $[a, b]$; the forcing set is $\mathcal{D}_L \subseteq \mathbb{R}^n$.

Our results: the greatest-element reduction, the two-cone decomposition and the descending-data criterion (Section 2, by one sign-chain lemma); the complete answer in order three, where the sharp constant is the reciprocal of a generalized trapezoidal weight (Section 3); the boundary characterization (Section 4); and the application to radial polyharmonic functions on annuli (Section 5). Background on disconjugacy and Chebyshev systems is in [1, 3, 8].

2. ONE CHAIN, THREE THEOREMS

We need the zero bound that gives disconjugate operators their name.

Lemma 2.1. *A nontrivial $u \in \mathcal{N}$ has at most $n-1$ zeros in $[a, b]$, counted with multiplicity.*

Proof. Induction on n ; for $n = 1$, $\mathcal{N}$ consists of the constants. Let $u \in \mathcal{N}$ have at least n zeros with multiplicity. If $L_1 u \equiv 0$ then u is constant, hence $\equiv 0$. Otherwise, by Rolle's theorem with multiplicities, u' has at least $n-1$ zeros with multiplicity, and so does $L_1 u = u'/w_1$ (the weight is positive and smooth). But $L_1 u$ lies in the kernel of the $(n-1)$ -st order tower built from $(w_2, \dots, w_{n-1})$, so by induction $L_1 u \equiv 0$, a contradiction. $\square$

Lemma 2.2 (Rigidity). *The data map $u \mapsto (L_0 u(a), \dots, L_{n-2} u(a); u(b))$ is a linear isomorphism $\mathcal{N} \rightarrow \mathbb{R}^n$. We write $v_d \in \mathcal{N}$ for the preimage of d .*

Proof. Both spaces have dimension n . Every $u \in \mathcal{N}$ expands as $u = \sum_{j=0}^{n-1} L_j u(a) u_j$ (both sides have equal quasi-derivatives at a , and elements of $\mathcal{N}$ with all quasi-derivatives zero at a vanish identically, by the expansion of $L_{n-1} u \equiv \text{const}$ through the tower). If u has zero data, then $u = L_{n-1} u(a) \cdot u_{n-1}$ and $0 = u(b) = L_{n-1} u(a) u_{n-1}(b)$ with $u_{n-1}(b) > 0$, so $u \equiv 0$. $\square$

The engine of the paper is the sign-chain lemma of [6, Theorem 4.1], transplanted to quasi-derivatives. Say that $h \in C[a, b]$ has *property (S)* if there is $\tau \in [a, b]$ with $h \leq 0$ on $[a, \tau]$ and $h \geq 0$ on $(\tau, b]$; and that h is *V-shaped* if it is nonincreasing on $[a, c]$ and nondecreasing on $[c, b]$ for some c .

Lemma 2.3 (Chain). *(i) Every nondecreasing h has (S). (ii) If $H \in C^1[a, b]$ and $H' = wG$ with $w > 0$ continuous and G having (S), then H is V-shaped; if moreover $H(a) \leq 0$, then H has (S).*

Proof. (i) $\{h < 0\}$ is an initial segment; take τ its supremum. (ii) H' has the sign pattern of G , so H is nonincreasing then nondecreasing. If also $H(a) \leq 0$: on $[a, \tau]$, $H \leq H(a) \leq 0$; on $[\tau, b]$, H is nondecreasing, so $\{H < 0\} \cap [\tau, b]$ is an initial segment whose supremum splits $[a, b]$ as required. $\square$

Theorem 2.4 (Greatest element; descending data). *Let $d \in \mathbb{R}^n$.*

- (i) $v_d \in \mathcal{K}_L(d)$, and $f \leq v_d$ on $[a, b]$ for every $f \in \mathcal{K}_L(d)$. Consequently $d \in \mathcal{D}_L$ if and only if $v_d \leq \max\{c_0, \beta\}$ on $[a, b]$.
- (ii) If $c_i \leq 0$ for $1 \leq i \leq n-2$, then every $f \in \mathcal{K}_L(d)$ is V-shaped; in particular such d is forcing for every value of β .
- (iii) $\mathcal{D}_L = \mathcal{A}_L \cup \mathcal{B}_L$, where $\mathcal{A}_L = \{d : v_d \leq c_0 \text{ on } [a, b]\}$ and $\mathcal{B}_L = \{d : v_d \leq \beta\}$ are closed convex cones.

Proof. (i) Membership: $L_{n-1}v_d$ is constant, hence nondecreasing, and by Lemma 2.2 the data match. For the bound, put $h = f - v_d$: then $L_{n-1}h$ is nondecreasing, $L_i h(a) = 0$ for $0 \leq i \leq n-2$, and $h(a) = h(b) = 0$. By Lemma 2.3(i), $L_{n-1}h$ has (S). Now descend through the indices $i = n-2, \dots, 1$: since $L_i h \in C^1$ satisfies $(L_i h)' = w_{i+1} L_{i+1} h$, Lemma 2.3(ii) together with $L_i h(a) = 0$ gives property (S) for $L_i h$. A final application of the first half of Lemma 2.3(ii) shows h is V-shaped; hence $h \leq \max\{h(a), h(b)\} = 0$. The forcing criterion follows as in [6]: if $v_d \leq \max\{c_0, \beta\}$ then every member is; conversely v_d is itself a member.

(ii) The same chain applied to f directly: $L_{n-1}f$ has (S); for $i = n-2, \dots, 1$, the hypothesis $L_i f(a) = c_i \leq 0$ propagates (S) down to $L_1 f$; then f is V-shaped, and a V-shaped function attains its maximum at an endpoint.

(iii) As in [6, Theorem 2.6]: the benchmark $\max\{c_0, \beta\}$ is a single number, equal to c_0 or β , and for fixed x the conditions $v_d(x) \leq c_0$, $v_d(x) \leq \beta$ are linear in d . $\square$

Remark 2.5. Specialized to $w_i \equiv 1$, we do not know a prior proof of (i) in this form, and even for $L = D^n$ it strikes us as simpler than both proofs on record: no integral representation, no differentiation under the integral—the monotone-remainder argument of [5] and [6, Lemma 2.1] is replaced by the same three-line chain that proves the descending-data criterion. That the two central sufficiency results of the theory are one lemma seems worth recording.

Remark 2.6. By Lemma 2.2 the forcing set $\mathcal{D}_L$ is linearly isomorphic to the endpoint-maximum cone $\mathcal{M}(\mathcal{N}) = \{u \in \mathcal{N} : \max_{[a,b]} u = \max\{u(a), u(b)\}\}$ of the kernel, exactly as $\mathcal{D}_n \cong \mathcal{M}_{n-1}$ in [6]. The two convex cones of Theorem 2.4(iii) are, by the same lemma, linear preimages of the cone of kernel elements nonnegative on $[a, b]$, whose structure is the classical subject of the Markov moment problem for Chebyshev systems [4].

3. ORDER THREE: THE SHARP CONSTANT IS A TRAPEZOID WEIGHT

For $n = 3$ the kernel is $\mathcal{N} = \text{span}\{1, u_1, u_2\}$, and the companion's sharp constant 2 generalizes as follows. Write $\Delta f = f(b) - f(a)$ and $t^+ = \max\{t, 0\}$.

Theorem 3.1. *Let $n = 3$. There are unique weights $\alpha, \beta_* \in \mathbb{R}$ with*

$$(1) \quad q(b) - q(a) = \alpha q'(a) + \beta_* q'(b) \quad \text{for all } q \in \mathcal{N},$$

namely

$$\alpha = \frac{\int_a^b w_1(x) \int_x^b w_2(t) dt dx}{w_1(a) \int_a^b w_2(t) dt} > 0, \quad \beta_* = \frac{\int_a^b w_1(x) \int_a^x w_2(t) dt dx}{w_1(b) \int_a^b w_2(t) dt} > 0.$$

Moreover $d = (c_0, c_1; \beta) \in \mathcal{D}_L$ if and only if

$$f'(a) = w_1(a) c_1 \leq \frac{1}{\alpha} (\beta - c_0)^+.$$

Proof. The identity. The functional $q \mapsto q(b) - q(a) - \alpha q'(a) - \beta_* q'(b)$ kills constants for any (α, β_*) , so (1) amounts to two linear conditions, exactness on u_1 and u_2 . Writing $W_1 = \int_a^b w_1$, $W_2 = \int_a^b w_2$, $B = \int_a^b w_1(x) \int_a^x w_2(t) dt dx$, these read

$$W_1 = \alpha w_1(a) + \beta_* w_1(b), \quad B = \beta_* w_1(b) W_2,$$

using $u'_1 = w_1$, $u'_2 = w_1 \int_a^\cdot w_2$. Solving: $\beta_* = B/(w_1(b)W_2) > 0$ and $\alpha = (W_1 - B/W_2)/w_1(a)$; since $W_1 W_2 - B = \int_a^b w_1(x) \int_x^b w_2 dt dx > 0$, also $\alpha > 0$, with the stated formulas.

The shape. By Theorem 2.4 we must decide when $q := v_d$ obeys the maximum principle. On $\mathcal{N}$, $L_2 q \equiv c$ is constant and $q' = w_1(c_1 + c \int_a^\cdot w_2)$. If $c \geq 0$, then q'/w_1 is nondecreasing, so q is V-shaped: the principle holds. If $c < 0$, then q'/w_1 is strictly decreasing, so q' has at most one sign change, from $+$ to $-$: the principle fails precisely when $q'(a) > 0 > q'(b)$. In every case,

$$\text{maximum principle fails} \iff q'(a) > 0 \text{ and } q'(b) < 0.$$

The criterion. By (1) and $\beta_* > 0$, given $q'(a) > 0$ we have $q'(b) < 0 \iff \beta - c_0 < \alpha q'(a)$. So failure is equivalent to $q'(a) > 0$ and $q'(a) > (\beta - c_0)/\alpha$, i.e. to $q'(a) > \alpha^{-1}(\beta - c_0)^+$; and $q'(a) = w_1(a)L_1 q(a) = w_1(a)c_1$. $\square$

Example 3.2. $L = D^3$ on $[0, 1]$: $w_1 = w_2 = 1$, $\alpha = \int_0^1 (1-x) dx = \frac{1}{2}$, and the criterion is $f'(a) \leq 2(f(b) - f(a))^+$: [6, Theorem 3.1]. The mysterious constant 2 is the reciprocal of the trapezoidal weight—the sharp threshold is governed by the quadrature rule that integrates the kernel exactly.

Example 3.3. $\mathcal{N} = \text{span}\{1, e^{\lambda x}, e^{\mu x}\}$ on $[0, 1]$ ($\lambda \neq \mu$, both nonzero): here $w_1 = e^{\lambda x}$, $w_2 = e^{(\mu-\lambda)x}$, and

$$\alpha = \frac{e^{\mu-\lambda} \frac{e^\lambda - 1}{\lambda} - \frac{e^\mu - 1}{\mu}}{e^{\mu-\lambda} - 1} \xrightarrow{\lambda, \mu \rightarrow 0} \frac{1}{2}.$$

For $(\lambda, \mu) = (1, -\frac{1}{2})$, $\alpha = 0.519441\dots$: the forcing threshold for this exponential family is $f'(0) \leq 1.925\dots \cdot (\Delta f)^+$, slightly below the polynomial constant 2.

4. THE BOUNDARY OF THE FORCING CONE

The boundary characterization of [6, Theorem 6.2] transplants once two bump functions are found in the kernel.

Lemma 4.1 (Bumps). *For $n \geq 3$ there exist $\psi_1, \psi_2 \in \mathcal{N}$, positive on (a, b) , vanishing at both endpoints, with ψ_1 vanishing at a to order exactly $n-2$ and at b to order one, and ψ_2 vanishing at a to order one and at b to order exactly $n-2$; in particular $\psi'_2(a) > 0 > \psi'_1(b)$.*

Proof. Let $\psi_1 \in \mathcal{N}$ have data $L_i \psi_1(a) = 0$ for $i \leq n-3$, $L_{n-2} \psi_1(a) = 1$, $\psi_1(b) = 0$ (Lemma 2.2). Its zero at a has order exactly $n-2$ and its zero at b order at least one, totalling at least $n-1$; by Lemma 2.1 there is no room for interior zeros or extra multiplicity, so $\psi_1 \neq 0$ on (a, b) , with sign determined near a by $L_{n-2} \psi_1(a) = 1$: positive. For ψ_2 , run the same argument from the right endpoint: take the kernel element vanishing simply at a , to order $n-2$ at b , normalized positive inside (the data map based at b is an isomorphism by the symmetric form of Lemma 2.2). $\square$

Theorem 4.2. *Let $n \geq 3$, let $\mathcal{M}(\mathcal{N}) = \{u \in \mathcal{N} : \max_{[a,b]} u = \max\{u(a), u(b)\}\}$, let $p \in \mathcal{M}(\mathcal{N})$, and let $B = \max\{p(a), p(b)\}$. Then $p \in \partial \mathcal{M}(\mathcal{N})$ if and only if (T) $p(m) = B$ for some $m \in (a, b)$, or (E) some endpoint at which B is attained is a critical point of p .*

Proof. As in [6, Theorem 6.2], with the following substitutions. For $(\Leftarrow)$ in case (T), perturb by $\varepsilon\psi_1$: endpoint values are unchanged and the value at m increases. In case (E) at the endpoint a , note $B - p(x) \leq C(x - a)^2$ (since $p(a) = B$, $p'(a) = 0$) and perturb by $\varepsilon\psi_2$: since $\psi_2(x) \geq c(x - a)$ near a with $c > 0$, the argument of [6] gives $p + \varepsilon\psi_2 \notin \mathcal{M}(\mathcal{N})$; at the endpoint b use ψ_1 , which vanishes simply there. For $(\Rightarrow)$, the compactness argument is verbatim: if neither (T) nor (E) holds, every kernel element C^1 -close to p (equivalently, close in data, by Lemma 2.2 and finite dimension) still lies in $\mathcal{M}(\mathcal{N})$. $\square$

Remark 4.3. The finer structure of [6]—the contact-type stratification, the connectedness of the four walls, and the closure poset—should transplant as well, with the polynomial factorization of $B - p$ replaced by the division theory of extended Chebyshev systems (prescribing zeros reduces to a lower-dimensional ECT system; see [8, §9.1] or [3]). We have verified the ingredients but not written the bookkeeping; we record it as Problem 6.1.

5. RADIAL POLYHARMONIC FUNCTIONS ON ANNULI

Fix $d \geq 1$, $0 < r_0 < R$, and let $\Delta_{\text{rad}}u = r^{1-d}(r^{d-1}u')'$ be the radial Laplacian of $\mathbb{R}^d$ on $[r_0, R]$. Then Δ_{rad}^m is, up to the positive factor r^{1-d} , the operator of our standing form of order $n = 2m$ with weights alternating between r^{1-d} and r^{d-1} :

$$L_1u = r^{d-1}u', \quad L_2u = \Delta_{\text{rad}}u, \quad L_3u = r^{d-1}(\Delta_{\text{rad}}u)', \quad \dots, \quad L_{2m-1}u = r^{d-1}(\Delta_{\text{rad}}^{m-1}u)',$$

all weights positive on $[r_0, R]$ since $r_0 > 0$, and $\Delta_{\text{rad}}^m 1 = 0$. Theorem 2.4 therefore applies verbatim.

Corollary 5.1. *Let $u \in C^{2m}(\bar{A})$ be a radial function on the closed annulus $\bar{A} = \{r_0 \leq |x| \leq R\} \subset \mathbb{R}^d$ with $\Delta^m u \geq 0$, and prescribe the data*

$$u(r_0), \quad L_1u(r_0), \quad \dots, \quad L_{2m-2}u(r_0); \quad u(R).$$

The class of all such u has a pointwise-greatest element: the radial m -polyharmonic function v with the same data. Consequently the data forces $u \leq \max\{u(r_0), u(R)\}$ on $\bar{A}$ if and only if the single function v satisfies it; and the data forces for every value of $u(R)$ whenever

$$L_i u(r_0) \leq 0 \quad \text{for } 1 \leq i \leq 2m - 2,$$

that is, whenever the quasi-derivatives $r^{d-1}\partial_r$ and Δ alternate nonpositively at the inner boundary.

For $m = 1$ this recovers the classical maximum principle for subharmonic functions (every data vector is forcing, as $\mathcal{K}$ consists of V-shaped functions); for $m = 2$, radial biharmonic comparison on annuli, the forcing set is the analogue of $\mathcal{D}_4$ of [6] for the kernel $\text{span}\{1, r^{2-d}, r^2, r^{4-d}\}$ (logarithms for $d \in \{2, 4\}$), and its complete description—the annular counterpart of the two curved lenses—is a concrete open problem. Nothing here uses the sign properties of Green's functions on balls; disconjugacy of the radial operator substitutes for Boggio's theorem, which is why annuli, where positivity preservation for the clamped plate can fail (see [2]), are nevertheless within reach for radial data.

6. PROBLEMS

Problem 6.1. Transplant the contact-type stratification, wall connectedness, and closure poset of [6] to $\mathcal{M}(\mathcal{N})$ for a general disconjugate L , using ECT division in place of polynomial factorization.

Problem 6.2. Order four: determine $\mathcal{D}_L$ completely for fourth-order disconjugate L . The reflection $x \mapsto a + b - x$, which halved the work for $L = D^4$ in [6], is no longer available (it does not preserve $\ker L$); a genuinely two-chart analysis is needed. The radial biharmonic annulus of Corollary 5.1 is the case to do first.

Problem 6.3. Overshoot: for a normalization of the data adapted to the canonical basis ($|c_i| \leq 1$), compute $E(L) = \sup\{\max f - \max\{f(a), f(b)\}\}$ and identify the extremal member, generalizing the truncated-exponential family and the limit constant $e - 1$ of [6].

Problem 6.4. Balls rather than annuli: as $r_0 \rightarrow 0^+$ the weights r^{1-d} degenerate and the data at the inner boundary collapses to derivatives at the center. Formulate the limiting forcing problem for radial polyharmonic functions on the ball, and connect it with the positivity theory of [2].

ACKNOWLEDGMENTS

This note is a companion to [6] and was prompted by the observation that none of the proofs there require polynomials.

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