FORCING TWO-POINT DATA FOR n-CONVEX SEQUENCES: THE DISCRETE FAMILY OF SPLITS

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ABSTRACT. Tenth in a series descending from [1]: we discretize the full family of two-point Hermite splits of [9], completing the matrix {continuous, discrete} $\times$ {all splits}. For sequences $x_0, \ldots, x_N$ with $\Delta^n x \geq 0$ and split data—the first p and last q values prescribed, $p+q=n$, equivalently the forward jet at 0 and backward jet at N—the two-point Lagrange interpolant is a pointwise-greatest element for q odd and least for q even, by a four-line divided-difference argument whose only input is the nonnegativity of n-th divided differences of n-convex sequences. Three of the continuous family's features survive with exact discrete forms, and one does not. The cardinal interpolants have closed falling-factorial forms with the *same* binomial coefficients as the continuum, $H_k(j) = \binom{j}{k} (N-j)^{\underline{q}} \sum_{i \leq p-1-k} \binom{q-1+i}{i} (j-k)^{\underline{i}}/(N-k)^{\underline{q+i}}$, manifestly nonnegative on the grid—the engine being the identity $\Delta_j \left[(N-j)^{\underline{q}}\right]^{-1} = q\left[(N-j)^{\underline{q+1}}\right]^{-1}$; consequently the sign criterion $(-1)^{q-k}(\text{outward jet})_k \geq 0$ forces at every split, containing the descending criterion of [4]. The worst-case overshoot is the maximum of one universal extremal, which at the splits $(2,1)$ and $(2,2)$ is the *same* discrete parabola $j(N-j)/(N-1)$; more generally $S_N^{(p,p+1)} \equiv S_N^{(p+1,p+1)}$, the near-balanced tie of [9], here by a node-counting proof shorter than the continuous one. What does *not* survive is curvature: for every split and every grid the forcing set is a union of two *polyhedral* cones—the "curvature is a continuum effect" principle of [4] holds uniformly across the family—and, a genuinely discrete twist, the continuum's unit-coefficient nonconvexity witnesses can fail on coarse grids and are repaired by extremal ratio constants.

1. INTRODUCTION

Two axes organize this series: the data pattern, and the underlying domain. Along the first axis, [9] unified all two-point Hermite splits (p,q) of $f^{(n)} \geq 0$ into one family with split-blind geometry and one ceiling law. Along the second, [4] discretized the one-sided pattern $(n-1,1)$ and found that the essential constants survive intact while the curved geometry does not. This note completes the square: all splits, on the grid.

The discrete split data is particularly natural: prescribing the forward difference jet $\Delta^k x_0$ $(0 \leq k \leq p-1)$ and the backward jet $\nabla^k x_N$ $(0 \leq k \leq q-1)$ is exactly prescribing the first p and the last q *values* of the sequence. The barrier is then the plain Lagrange interpolant through the n prescribed points, and the comparison sign comes from a divided-difference argument (Section 3) that replaces both the Green functions of [9] and the sign-chain of [4]: an n-convex sequence has nonnegative n-th divided differences over *arbitrary* grid nodes—a classical fact of higher-order convexity, with a six-line proof included—and evaluating one well-chosen divided difference gives the sign $(-1)^q$ at every free point at once.

Highlights:

- **Exact discrete cardinals, same binomial coefficients** (Theorem 5.1). The identity $\Delta_j \left[1/(N-j)^{\underline{q}}\right] = q/(N-j)^{\underline{q+1}}$ makes the Newton coefficients of the cardinal interpolants

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explicit: the falling-factorial forms display them as products of nonnegative factors, with the continuum's $\binom{q-1+i}{i}$ untouched. Positivity of the basis gives the family's sign criterion $(-1)^{q-k}(\text{outward jet})_k \geq 0$ at every split (Theorem 5.2), specializing at $q = 1$ to the descending criterion of [4, Theorem 2.3].

- **Polyhedrality across the family** (Theorem 4.1). Each chart of the forcing set is cut out by at most N linear inequalities; the union is a pair of polyhedral cones for every split, every order, and every grid. The curvature that enters the continuous theory at order four—for every split, by [9, Theorem 3.1]—is a pure continuum effect, uniformly in the split. After rescaling, the charts should converge to the continuum cones on normalized sections (Problem 7.2).
- **The discrete parabola returns, and the tie discretizes** (Theorem 6.1, Theorem 6.2). The overshoot extremal at split $(2, 2)$ is *identically* the extremal $j(N-j)/(N-1)$ of [4, Theorem 5.1] at split $(2, 1)$: the discrete incarnation of the $x(1-x)$ coincidence that has followed this series since [2]. In general $S_N^{(p,p+1)} \equiv S_N^{(p+1,p+1)}$ on every grid—the near-balanced tie of [9, Proposition 5.2]—with a proof by counting nodes that is shorter than the continuous Pascal-step argument.
- **A genuinely discrete correction** (Example 4.2). The continuum's nonconvexity witnesses, with unit data coefficients, can fail on coarse grids (already at $(p, q, N) = (2, 3, 6)$); extremal ratio constants repair them at every split and grid.

Section 2 sets the stage, Section 3 proves the reduction, Section 4 the polyhedral geometry, Section 5 the cardinals and the sign criterion, Section 6 the overshoot, tie, and continuum limit; Section 7 collects problems. All claims are verified in exact rational arithmetic; the script accompanies the source.

2. Setting

Fix $n \geq 2$, a split (p, q) with $p, q \geq 1$, $p + q = n$, and a grid $\{0, 1, \dots, N\}$ with $N \geq n$. For $D = (\alpha; \beta) = (\alpha_0, \dots, \alpha_{p-1}; \beta_0, \dots, \beta_{q-1})$ let

$$\mathcal{K}(D) = \big\{x \in \mathbb{R}^{N+1} : \Delta^n x \geq 0,\quad \Delta^k x_0 = \alpha_k\ (k < p),\quad \nabla^k x_N = \beta_k\ (k < q)\big\},$$

where $\Delta x_j = x_{j+1} - x_j$, $\nabla x_j = x_j - x_{j-1}$, and $\Delta^n x \geq 0$ means $\Delta^n x_j \geq 0$ for $0 \leq j \leq N - n$. Since $x_j = \sum_k \binom{j}{k} \Delta^k x_0$ and $x_{N-l} = \sum_k \binom{l}{k}(-1)^k \nabla^k x_N$, the data prescribes exactly the values $x_0, \dots, x_{p-1}$ and $x_{N-q+1}, \dots, x_N$. As throughout the series, D is *max-forcing* if every member satisfies $x_j \leq \max\{x_0, x_N\}$ for all j, *min-forcing* with $\geq$ min, both discussed for nonempty classes; the class is never empty (the barrier below belongs to it). Let v_D be the unique polynomial sequence of degree $\leq n - 1$ through the n prescribed values: Lagrange interpolation at the *split nodes*

$$T \;=\; \{0, \dots, p-1\} \cup \{N - q + 1, \dots, N\}.$$

The reflection $\rho x_j := (-1)^n x_{N-j}$ preserves the class and exchanges the splits $(p, q) \leftrightarrow (q, p)$, exactly as in [9, §2].

3. The reduction

For distinct reals $t_0, \dots, t_n$ write $[t_0, \dots, t_n]f = \sum_i f(t_i)/\omega'(t_i)$, $\omega(t) = \prod_i (t - t_i)$, for the divided difference; it satisfies, for any two designated nodes $a \neq b$ of a set $X \cup \{a, b\}$,

$$[X \cup \{a, b\}]f \;=\; \frac{[X \cup \{b\}]f - [X \cup \{a\}]f}{b - a}. \tag{1}$$

Lemma 3.1 (n-convexity at arbitrary nodes)**.** *If $\Delta^n x_j \geq 0$ for all admissible j, then $[S]x \geq 0$ for every set S of $n + 1$ grid points.*

Proof. Induct on the diameter $t_n - t_0$ of $S = \{t_0 < \cdots < t_n\}$. If $t_n - t_0 = n$, the nodes are consecutive and $[S]x = \Delta^n x_{t_0}/n! \geq 0$. Otherwise pick a grid point $u \in (t_0, t_n) \setminus S$. Applying (1) to $S \cup \{u\}$ with the pairs (t_0, u) and (u, t_n) and eliminating $[S \cup \{u\}]$,

$$[S]x \;=\; \frac{t_n - u}{t_n - t_0}\,\big[(S \setminus \{t_0\}) \cup \{u\}\big]x \;+\; \frac{u - t_0}{t_n - t_0}\,\big[(S \setminus \{t_n\}) \cup \{u\}\big]x :$$

a convex combination of divided differences over node sets of strictly smaller diameter, both ≥ 0 by induction. □

Theorem 3.2 (Reduction, all splits)**.** *Let q be odd. Then v_D is the pointwise-greatest element of $\mathcal{K}(D)$; D is max-forcing iff $v_D(j) \leq \max\{x_0, x_N\}$ for all j; and no data of this split is min-forcing. For q even: least element, min-forcing iff $v_D \geq \min\{x_0, x_N\}$, and no data is max-forcing.*

Proof. Let $x \in \mathcal{K}(D)$ and $h = x - v_D$, which vanishes on T and has $\Delta^n h \geq 0$. For a free index $j \notin T$, Lemma 3.1 applied to $S = T \cup \{j\}$ gives, since h vanishes on T,

$$0 \;\leq\; [S]h \;=\; \frac{h_j}{\prod_{t \in T}(j - t)},$$

and the product has sign $(-1)^q$ (p positive factors from the left block, q negative from the right). Hence $(-1)^q h_j \geq 0$ for every j: greatest element for q odd, least for q even, and the barrier criterion follows since $v_D \in \mathcal{K}(D)$. For the no-go: the map $x \mapsto (\text{split data}, \Delta^n x)$ is a linear bijection of $\mathbb{R}^{N+1}$ onto $\mathbb{R}^n \times \mathbb{R}^{N-n+1}$ (dimensions match, and a polynomial sequence of degree $\leq n-1$ vanishing at the n split nodes is zero), so for each s there is $w^{(s)}$ with zero split data and $\Delta^n w^{(s)} = e_s$; it is nonzero, one-signed with sign $(-1)^q$ by the above, hence $(-1)^q w^{(s)} > 0$ at some free index, and $x + t\,w^{(s)}$ $(t \geq 0)$ stays in the class while escaping in the direction the opposite principle would need to control. □

4. Polyhedral geometry, at every split

Theorem 4.1 (Structure)**.** *Let q be odd (mirror for q even). The forcing set is $\mathcal{R} \cup \mathcal{L}$ with $\mathcal{R} = \{D : v_D \leq x_N$ on the grid$\}$ and $\mathcal{L} = \{D : v_D \leq x_0\}$, and:*

(i) *with $\chi_D(j) := (v_D(j) - x_N)/(j - N)$, a polynomial of degree $\leq n-2$ in j depending linearly on D, membership in $\mathcal{R}$ is equivalent to the N linear inequalities $\chi_D(j) \geq 0$, $0 \leq j \leq N-1$; likewise for $\mathcal{L}$;*

(ii) *each chart is a polyhedral convex cone, for every split, order, and grid; the forcing set is a union of two polyhedral cones, extending* [4, Theorem 2.3(iv)] *to the whole family, in contrast to the continuum, where no chart is polyhedral beyond order three for any split* [9, Theorem 3.1]*;*

(iii) *for $p, q \geq 2$ the forcing set is not convex.*

Proof. (i) $v_D - x_N$ vanishes at $j = N$, so the division is exact; $j - N < 0$ on the rest of the grid flips the inequality; the conditions at the right nodes $j \in \{N-q+1, \ldots, N-1\}$ read $\chi_D(j) = (\text{prescribed value} - x_N)/(j - N)$, linear in the data, and the condition at $j = 0$ contains the dominance $x_0 \leq x_N$. (ii) Finitely many linear inequalities. (iii) Let H_k denote the cardinal barriers of Section 5 and set

$$c \;:=\; \min\big\{H_0(j)/H_1(j) : H_1(j) > 0\big\} \;>\; 0,$$

a minimum of finitely many positive ratios ($H_0 > 0$ wherever $H_1 > 0$, by the closed forms). The data a with $\alpha_1 = c$, $\beta_0 = 1$, all else zero, satisfies $v_a - 1 = cH_1 - H_0 \leq 0$, so a forces with $\Delta x_0 = c > 0$; symmetrically b with $\alpha_0 = 1$, $\beta_1 = -c'$ forces with $\nabla x_N < 0$. Their midpoint has $x_0 = x_N$, $\Delta x_0 = c/2 > 0$, and $\nabla x_N = -c'/2 < 0$: it violates the necessary conditions of both charts—$\chi_{\mathcal{R}}(N-1) = \nabla x_N \ge 0$ and, mirrored, $\Delta x_0 \le 0$—at a benchmark tie, so it does not force. (For q even the mirrored witnesses work; both parities are verified in exact arithmetic in the script.) $\square$

Example 4.2 (The continuum witness can fail on a coarse grid). In the continuum, the witness has unit coefficient: $H_{10} \le H_{00}$ termwise [9, Theorem 3.1], so $\alpha_1 = 1$ works. Discretely this can fail: at $(p,q,N) = (2,3,6)$ one computes $H_1(3) = 0.3 > 0.2 = H_0(3)$, so the unit datum does *not* force, and the extremal ratio $c = \min H_0/H_1 = \frac{2}{3} < 1$ is genuinely needed. On the grid, the safe inward slope at one end depends on the grid, not only on the split.

Remark 4.3. As $N \to \infty$ with the split fixed, the chart of (i), read in the rescaled variable $u = j/N$ and rescaled data (Proposition 6.3), should converge to the continuum chart $\{\chi \ge 0 \text{ on } [0,1]\} \cong \mathcal{P}_{n-2}$ of [9, Theorem 3.1]: the polyhedral walls densify into the Markov–Lukács boundary. Proposition 6.3 proves uniform convergence of the cardinal maps and overshoot extremals, not of the cones themselves—the global Hausdorff distance between distinct unbounded cones is generally infinite, so the correct convergence statement is about normalized compact sections. Making that mode precise is part of Problem 7.2, exactly as in [4, Remark 2.4].

5. Cardinals and the sign criterion

Write $a^{\underline{i}} = a(a-1)\cdots(a-i+1)$ for falling factorials. The engine of this section is the one-line identity

$$\Delta_j \frac{1}{(N-j)^{\underline{q}}} = \frac{1}{(N-1-j)^{\underline{q}}} - \frac{1}{(N-j)^{\underline{q}}} = \frac{(N-j)-(N-j-q)}{(N-j)^{\underline{q+1}}} = \frac{q}{(N-j)^{\underline{q+1}}}, \tag{2}$$

valid where the denominators are nonzero, i.e. for $j \le N-q-1$; the Newton expansion below uses it only at $j = k, \ldots, p-1 \le N-q-1$, inside that range by the standing assumption $N \ge n$.

Theorem 5.1 (Discrete cardinals)**.** *For $0 \le k \le p-1$, the (p,q) cardinal H_k (the barrier of the datum $\Delta^k x_0 = 1$, all other data zero) is*

$$H_k(j) = \binom{j}{k} (N-j)^{\underline{q}} \sum_{i=0}^{p-1-k} \binom{q-1+i}{i} \frac{(j-k)^{\underline{i}}}{(N-k)^{\underline{q+i}}},$$

which is nonnegative on the whole grid; the right cardinals are $H_k^{\nabla}(j) = (-1)^k H_k^{(q,p)}(N-j)$, one-signed with sign $(-1)^k$; and $H_0 + H_0^{\nabla} \equiv 1$. In the limit $j = \lfloor uN \rfloor$, $N \to \infty$, one has $N^{-k} H_k(j) \to H_{k0}^{\mathrm{cont}}(u)$, the continuum cardinal of [9, Lemma 4.1]*, with the same coefficients $\binom{q-1+i}{i}$.*

Proof. Write $\varphi(j) = 1/(N-j)^{\underline{q}}$. By (2) and induction, $\Delta^i \varphi(j) = q(q+1)\cdots(q+i-1)\,/(N-j)^{\underline{q+i}}$, so the Newton expansion of φ at the consecutive nodes $k, k+1, \ldots, p-1$ is

$$G(j) = \sum_{i=0}^{p-1-k} \frac{\Delta^i \varphi(k)}{i!} (j-k)^{\underline{i}} = \sum_{i=0}^{p-1-k} \binom{q-1+i}{i} \frac{(j-k)^{\underline{i}}}{(N-k)^{\underline{q+i}}},$$

the interpolant of φ at those nodes. The displayed $H_k = \binom{j}{k}(N-j)^{\underline{q}} G(j)$ then has degree $\le k + q + (p-1-k) = n-1$; it vanishes at the right block ($(N-j)^{\underline{q}} = 0$ there) and at $j = 0, \ldots, k-1$ ($\binom{j}{k} = 0$); and at $j = k, \ldots, p-1$ it equals $\binom{j}{k}(N-j)^{\underline{q}} \varphi(j) = \binom{j}{k}$, which is exactly the value pattern of the datum $\Delta^k x_0 = 1$. Interpolation at n nodes is unisolvent, so this *is* the cardinal. Nonnegativity: on the grid every factor is ≥ 0—$\binom{j}{k} \ge 0$; $(N-j)^{\underline{q}} \ge 0$ for $j \le N$ (zero on the right block); and each $(j-k)^{\underline{i}} \ge 0$ wherever $\binom{j}{k} \ne 0$ (for $j \ge k$ a falling factorial of a nonnegative integer). The reflection and partition statements follow from ρ and uniqueness (the all-ones sequence interpolates the data of $H_0 + H_0^{\nabla}$). The limit is immediate from $\binom{j}{k} \sim (uN)^k/k!$, $(N-j)^{\underline{q}} \sim ((1-u)N)^q$, $(j-k)^{\underline{i}} \sim (uN)^i$, $(N-k)^{\underline{q+i}} \sim N^{q+i}$. □

Theorem 5.2 (Sign criterion, all splits)**.** *Define the outward jets $\partial_{\mathrm{out}}^k x(0) = (-1)^k \Delta^k x_0$ and $\partial_{\mathrm{out}}^k x(N) = \nabla^k x_N$. If*

$$(-1)^{q-k}\,\partial_{\mathrm{out}}^k x \;\geq\; 0 \quad \textit{at both ends} \qquad (1 \leq k \leq p-1 \textit{ at } 0, \quad 1 \leq k \leq q-1 \textit{ at } N),$$

then D forces (max- for q odd, min- for q even), for arbitrary x_0, x_N. At $q = 1$ this is the descending criterion $\Delta^k x_0 \leq 0$ of [4, Theorem 2.3(ii)]*; the pattern is that of* [9, Theorem 4.3], *verbatim.*

Proof. Identical to [9, Theorem 4.3]: expand the barrier in the cardinal basis using $H_0 + H_0^{\nabla} = 1$; the hypothesis makes every term one-signed in the direction of the principle, by Theorem 5.1. □

6. Overshoot: the discrete parabola, the tie, the limit

Define $E^{(p,q)}(N)$ as in [4, 9]: the largest violation of the principle over members with $|\Delta^k x_0| \leq 1$, $|\nabla^k x_N| \leq 1$ $(k \geq 1)$, endpoint values free.

Theorem 6.1 (Overshoot)**.** *$E^{(p,q)}(N) = \max_j S_N^{(p,q)}(j)$, where*

$$S_N^{(p,q)}(j) \;=\; \sum_{k=1}^{p-1} H_k(j) \;+\; \sum_{k=1}^{q-1} H_k^{(q,p)}(N-j)$$

is, up to the parity sign $(-1)^{q+1}$, the barrier of the universal data $\alpha_k = (-1)^{q+1}$, $\beta_k = (-1)^{q+k+1}$. In particular

$$S_N^{(2,1)}(j) \;=\; S_N^{(2,2)}(j) \;=\; \frac{j\,(N-j)}{N-1}, \qquad E^{(2,1)}(N) = E^{(2,2)}(N) = \frac{\lfloor N^2/4 \rfloor}{N-1} \qquad (N \geq 4):$$

the discrete parabola of [4, Theorem 5.1] *is also the worst case of the discrete clamped split (the range $N \geq 4$ puts both the order-three and the order-four class inside the standing hypothesis).*

Proof. The bound, attainment, and universal-extremal identity are word for word as in [9, Theorem 5.1], with Theorem 5.1 supplying positivity, reflection, and the partition of unity. The two closed forms: at $(2,1)$, the single cardinal is $H_1(j) = j(N-j)/(N-1)$ directly from Theorem 5.1; at $(2,2)$, $S = H_1(j) + H_1^{(2,2)}(N-j)$ with $H_1^{(2,2)}(j) = j\,(N-j)^{\underline{2}}/(N-1)^{\underline{2}}$, and

$$\frac{j(N-1-j)(N-j) + (N-j)(j-1)\,j}{(N-1)(N-2)} \;=\; \frac{j(N-j)\,(N-2)}{(N-1)(N-2)} \;=\; \frac{j(N-j)}{N-1}. \qquad \square$$

Theorem 6.2 (The tie discretizes)**.** *For every $p \geq 1$ and every $N \geq 2p+1$ the polynomial identity*

$$S_N^{(p,\,p+1)} \;\equiv\; S_N^{(p+1,\,p+1)}$$

holds on the grid; consequently $E^{(p,p+1)}(N) = E^{(p+1,p)}(N) = E^{(p+1,p+1)}(N)$ for every $N \geq 2p+2$ (the range where all three classes satisfy the standing assumption $N \geq n$): the near-balanced tie of [9, Proposition 5.2]*, on every grid. At $N = 2p+1$ the balanced side sits outside the standing assumption, but its cardinal interpolants remain defined—the $2p+2$ split nodes are then exactly the gridpoints, still distinct—and the identity is asserted at that level; no class or overshoot claim is made there.*

Proof. By counting nodes. Both sides are polynomials in j. The balanced side has degree $\le 2p{+}1$ but is symmetric by construction (its two sums exchange under $j \mapsto N-j$), hence of even degree; so both sides have degree $\le 2p$. They agree at the $2p+1$ points $\{0,\dots,p-1\}\cup\{N-p,\dots,N\}$. At $j \le p-1$: the right-hand sums of both splits vanish ($N-j$ lies in the right node block of the reflected split in both cases), and the left sums both equal $\sum_{k=1}^{p-1}\binom{j}{k}$—the balanced side's extra cardinal contributes $\binom{j}{p}=0$. At $j \ge N-p$: both left sums vanish (those points lie in both splits' right node blocks), and both right sums equal $\sum_{k=1}^{p}\binom{N-j}{k}$, the evaluation at the left nodes of the reflected splits. Two polynomials of degree $\le 2p$ agreeing at $2p+1$ points are identical. (The two blocks are disjoint since $N \ge 2p$.) $\square$

Proposition 6.3 (Continuum limit)**.** *Fix the split (p,q) and rescale the data bounds to $|\Delta^k x_0| \le N^{-k}$, $|\nabla^k x_N| \le N^{-k}$. Then the rescaled overshoot converges to the continuum value $E^{(p,q)}$ of* [9, Theorem 5.1]*:*

$$\max_j \sum_k N^{-k}\Big[H_k(j) + H_k^{(q,p)}(N-j)\Big] \;\longrightarrow\; E^{(p,q)} \qquad (N\to\infty).$$

In grid units (bounds $\equiv 1$) and for $\max(p,q)\ge 2$, the overshoot instead grows like $N^{\max(p,q)-1}$, the top jet dominating, exactly as observed for $(n-1,1)$ in [4, Remark 5.2]*; for the split $(1,1)$ there are no non-value data and $E^{(1,1)}(N)=0$.*

Proof. By Theorem 5.1, $N^{-k}H_k(\lfloor uN\rfloor) \to H^{\mathrm{cont}}_{k0}(u)$ uniformly on $[0,1]$ (ratios of falling factorials converge uniformly, and all terms are polynomials of bounded degree), so the rescaled extremal converges uniformly to $S^{(p,q)}$ and maxima converge. The grid-unit statement follows from the same scaling read in reverse. $\square$

7. Problems

Problem 7.1. The joint ceiling. For $n, N \to \infty$ with $p/n \to \lambda$ and the continuum rescaling of Proposition 6.3, in what regime of (n,N) does the discrete overshoot approach the ceiling $e^{\max(\lambda,1-\lambda)}-1$ of [9, Theorem 6.2]? The two limits do not obviously commute; the grid must resolve the moderate-deviation window near the transition, suggesting a threshold $N \gg n^{?}$.

Problem 7.2. Facets. Of the N inequalities cutting each chart, how many are facets? For $(n-1,1)$ this is [4, Question 6.1]; the split structure adds the question of how the facet count depends on (p,q) at fixed n, and how the irredundant systems converge to the Markov–Lukács boundary.

Problem 7.3. Weighted grids. Transplant the family to disconjugate difference operators (discrete quasi-derivatives with weights), fusing this note with [3]: does the divided-difference argument survive with weighted divided differences, and do the extremal ratio constants of Theorem 4.1(iii) admit uniform lower bounds?

Problem 7.4. Discrete centrality. On the balanced split, is $S_N^{(m,m)}$ maximized at the central gridpoint(s) for every m and N? The continuum proof [10] uses a derivative identity with no immediate difference analogue; small cases suggest yes.

Acknowledgments

Tenth in a series descending from [1], after [2, 3, 4, 5, 6, 7, 8, 9, 10]; it discretizes the family of [9] and extends [4] to all splits. All claims were verified in exact rational arithmetic.

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