# WHICH ENDPOINT DATA FORCE THE MAXIMUM PRINCIPLE FOR HIGHER-ORDER CONVEX SEQUENCES?

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Abstract. This is the discrete installment of [2, 3]: for sequences $x_0, \dots, x_N$ with nonnegative $n$-th differences, we determine which data $\Delta^i x_0$ $(0 \le i \le n-2)$, $x_N$ force the maximum principle $x_j \le \max\{x_0, x_N\}$. The sign-chain argument of [3] transfers verbatim: the class has a greatest element (the interpolating sequence that is polynomial of degree $\le n-1$), forcing is decided by that single sequence, descending data $\Delta^i x_0 \le 0$ force unconditionally, and the forcing set is a union of two convex cones. For $n = 3$ the sharp criterion is $\Delta x_0 \le \frac{2}{N}(x_N - x_0)^+$—the continuous constant with $b - a = N$ and, perhaps surprisingly, no discretization correction. The maximal overshoot under $|\Delta^i x_0| \le 1$ is computed exactly; for $n = 3$ it is $\lfloor N^2/4 \rfloor/(N-1)$. Two phenomena are genuinely discrete. The boundary of the forcing cone has a single mechanism: the endpoint-critical stratum of the continuous theory merges into the tangency stratum, since a critical endpoint on the grid *is* an interior tie. And the forcing set is a finite union of two *polyhedral* cones for every $n$ and $N$—so the curved walls that appear in the continuous theory at $n = 4$ are a pure continuum effect: on the grid every wall is polygonal, with the parabolic wall expected in the refinement limit.

## 1. Introduction

Fix integers $n \ge 2$ and $N \ge n - 1$ (when $N = n - 1$ the data below determine the entire sequence and every statement holds trivially; nothing is lost by picturing $N \ge n$). For a sequence $x = (x_0, \dots, x_N) \in \mathbb{R}^{N+1}$ write $\Delta x_j = x_{j+1} - x_j$ and iterate. The discrete class corresponding to $f^{(n)} \ge 0$ is

$$\Delta^n x_j \;\ge\; 0 \qquad (0 \le j \le N - n),$$

the *convex sequences of order* $n$ (see, e.g., [4]); its members satisfy the maximum principle $x_j \le \max\{x_0, x_N\}$ for $n \le 2$ but not in general for $n \ge 3$. Following [2], prescribe the data

$$d \;=\; \big(c_0, c_1, \dots, c_{n-2};\, \beta\big), \qquad c_i = \Delta^i x_0, \quad \beta = x_N,$$

write $\mathcal{K}(d)$ for the class of sequences with $\Delta^n x \ge 0$ and data $d$, call $d$ *forcing* if every member of $\mathcal{K}(d)$ satisfies the maximum principle, and let $\mathcal{D}(n, N) \subseteq \mathbb{R}^n$ be the set of forcing data.

The kernel of $\Delta^n$ is the space $\mathcal{N}$ of sequences polynomial of degree $\le n - 1$ in $j$, with the binomial basis $\binom{j}{i}$, $0 \le i \le n - 1$: indeed $\Delta^i \binom{j}{k}\big|_{j=0} = \delta_{ik}$. The theory of [2, 3] transfers to this setting essentially without friction—Section 2—and the sharp constant for $n = 3$ comes out with no discretization correction (Section 3), as does an exact overshoot formula (Section 5). What makes the discrete case worth a separate note are two structural simplifications with no continuous counterpart: the boundary of the forcing cone has a single boundary mechanism (Section 4), and the forcing set is a finite union of two *polyhedral* cones in every order (Theorem 2.3(iv)), which locates the curvature discovered in [2] precisely at the continuum limit.

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## 2. The core theory

**Lemma 2.1** (Rigidity). *For $N \geq n-1$ the map sending $p \in \mathcal{N}$ to its data*

$$\big(\Delta^0 p(0),\ \ldots,\ \Delta^{n-2}p(0);\ p(N)\big)$$

*is a linear isomorphism onto $\mathbb{R}^n$. Explicitly, the preimage of $d$ is*

$$v_d(j) \ = \ \sum_{i=0}^{n-2} c_i \binom{j}{i} \ + \ \frac{\beta - \sum_{i=0}^{n-2} c_i \binom{N}{i}}{\binom{N}{n-1}} \binom{j}{n-1}.$$

*Proof.* Expanding $p = \sum_{i=0}^{n-1} a_i \binom{j}{i}$, the data reads $(a_0, \ldots, a_{n-2}; \sum_i a_i \binom{N}{i})$, and $\binom{N}{n-1} \neq 0$. $\square$

Say a finite sequence $g$ has *property (S)* if there is an index $t$ with $g_j \leq 0$ for $j < t$ and $g_j \geq 0$ for $j \geq t$; and that $h$ is *V-shaped* if it is nonincreasing up to some index and nondecreasing thereafter.

**Lemma 2.2** (Chain). *(i) Every nondecreasing sequence has (S). (ii) If $\Delta H$ has (S), then $H$ is V-shaped; if moreover $H_0 \leq 0$, then $H$ has (S).*

*Proof.* (i) Take $t =$ the first index with $g_t \geq 0$ (or the length if none). (ii) $H$ is nonincreasing while $\Delta H \leq 0$ and nondecreasing thereafter; if $H_0 \leq 0$, then $H \leq 0$ on the nonincreasing stretch, and on the nondecreasing stretch $\{H < 0\}$ is an initial segment. $\square$

**Theorem 2.3.** *Let $n \geq 2$, $N \geq n-1$, $d \in \mathbb{R}^n$.*

- (i) *$v_d \in \mathcal{K}(d)$ and $x \leq v_d$ (entrywise) for every $x \in \mathcal{K}(d)$. Consequently $d \in \mathcal{D}(n, N)$ if and only if $v_d(j) \leq \max\{c_0, \beta\}$ for $0 \leq j \leq N$.*
- (ii) *If $c_i \leq 0$ for $1 \leq i \leq n-2$, then every $x \in \mathcal{K}(d)$ is V-shaped; such $d$ is forcing for every $\beta$.*
- (iii) *$\mathcal{D}(n, N) = \mathcal{A} \cup \mathcal{B}$ with $\mathcal{A} = \{d : v_d(j) \leq c_0\ \forall j\}$ and $\mathcal{B} = \{d : v_d(j) \leq \beta\ \forall j\}$ closed convex cones.*
- (iv) *Each of $\mathcal{A}, \mathcal{B}$ is cut out by $N$ linear inequalities: the $N-1$ conditions at the interior gridpoints, together with the endpoint-dominance inequality ($\beta \leq c_0$ for $\mathcal{A}$, $c_0 \leq \beta$ for $\mathcal{B}$); in particular $\mathcal{D}(n, N)$ is a finite union of two polyhedral cones, for every $n$ and $N$.*

*Proof.* (i) $v_d$ has $\Delta^n v_d \equiv 0$ and the right data. For $x \in \mathcal{K}(d)$ put $h = x - v_d$: then $\Delta^{n-1}h$ is nondecreasing (its difference is $\Delta^n x \geq 0$), $\Delta^i h_0 = 0$ for $i \leq n-2$, and $h_0 = h_N = 0$. Lemma 2.2(i) gives (S) for $\Delta^{n-1}h$; descending through $i = n-2, \ldots, 1$, Lemma 2.2(ii) with $\Delta^i h_0 = 0$ gives (S) for each $\Delta^i h$; a final application shows $h$ V-shaped, whence $h \leq \max\{h_0, h_N\} = 0$. The forcing criterion follows since $v_d$ is itself a member. (ii) The same chain applied to $x$ directly, using $\Delta^i x_0 = c_i \leq 0$. (iii) As in [2, Theorem 2.6]: the benchmark equals $c_0$ or $\beta$, and each condition $v_d(j) \leq c_0$, $v_d(j) \leq \beta$ is linear in $d$. (iv) The condition at $j = 0$ is trivial, while the condition at $j = N$ reads $v_d(N) = \beta \leq c_0$ (for $\mathcal{A}$; symmetrically $c_0 \leq \beta$ for $\mathcal{B}$): exactly the dominance inequality. Each of the finitely many conditions is a closed half-space. $\square$

*Remark* 2.4. Point (iv) is where the discrete theory parts company with the continuous one. In [2] it is proved that $\mathcal{D}_n$ is *not* a finite union of convex polyhedra for any $n \geq 4$: its boundary carries parabolic tangency walls. On the grid those walls are polygonal—finitely many linear conditions can express everything—and the curvature of the continuous forcing set is revealed as a pure continuum effect. Under grid refinement the polygonal walls should converge, after the natural rescaling of the difference data, to the parabolic wall of [2, Proposition 5.1]; making the mode of convergence precise, and counting facets, is Question 6.1.

## 3. Order three: the constant survives discretization intact

**Theorem 3.1.** *Let $n = 3$, $N \geq 2$, $d = (c_0, c_1; \beta)$. Then*

$$d \in \mathcal{D}(3, N) \quad \Longleftrightarrow \quad \Delta x_0 = c_1 \;\leq\; \frac{2}{N}\,(\beta - c_0)^+.$$

*Proof.* By Theorem 2.3 we decide the maximum principle for the quadratic sequence $q = v_d$, for which $\Delta q_j = a_1 + a_2 j$ is monotone ($a_2 = \Delta^2 q$ is constant). Summing the arithmetic progression gives the *discrete trapezoidal identity*

$$q_N - q_0 \;=\; \sum_{j=0}^{N-1} \Delta q_j \;=\; \frac{N}{2}\,\big(\Delta q_0 + \Delta q_{N-1}\big), \tag{1}$$

exact on quadratic sequences. If $a_2 \geq 0$, then $\Delta q$ is nondecreasing, $q$ is V-shaped, and the principle holds; moreover the stated inequality holds, since for $c_1 > 0$ monotonicity gives $\Delta q_{N-1} \geq \Delta q_0 = c_1$, whence (1) yields $\beta - c_0 \geq N c_1 \geq \frac{N}{2}\,c_1$. If $a_2 < 0$, then $\Delta q$ is strictly decreasing, and the principle fails precisely when $\Delta q_0 > 0 > \Delta q_{N-1}$: in that case the maximal entry $q_{j^*}$ satisfies $q_{j^*} \geq q_1 > q_0$ and $q_{j^*} \geq q_{N-1} > q_N$; while if $\Delta q_0 \leq 0$ the sequence is nonincreasing, and if $\Delta q_{N-1} \geq 0$ it is nondecreasing. By (1), given $\Delta q_0 > 0$, the condition $\Delta q_{N-1} < 0$ is equivalent to $\beta - c_0 < \frac{N}{2}\,\Delta q_0$; packaging with the positive part as in [2, Theorem 3.1] finishes the proof. ☐

*Remark* 3.2. The constant is exactly the continuous one for an interval of length $N$: no integer-rounding correction appears, even though the vertex of the extremal parabola generally falls between gridpoints. The reason is visible in the proof: the failure condition "rises at the first step and falls at the last" and the trapezoidal identity are both *exact* on the grid; nothing about the criterion ever looks between gridpoints. The same is not expected for $n \geq 4$ (Question 6.1).

## 4. The boundary has one mechanism

Let $\mathcal{M}(n-1, N) = \{p \in \mathcal{N} : \max_{0 \leq j \leq N} p(j) = \max\{p(0), p(N)\}\}$, the polyhedral model of $\mathcal{D}(n, N)$ under Lemma 2.1.

**Theorem 4.1.** *Let $n \geq 3$, $N \geq 3$, $p \in \mathcal{M}(n-1, N)$, and $B = \max\{p(0), p(N)\}$. Then $p \in \partial\mathcal{M}(n-1, N)$ if and only if $p(j) = B$ for some interior gridpoint $0 < j < N$. A nonconstant $p$ has at most $n-1$ ties with $B$ in $\{0, \ldots, N\}$ altogether.*

*Proof.* ($\Leftarrow$) The quadratic sequence $b(j) = j(N-j)$ lies in $\mathcal{N}$ (here $n \geq 3$), vanishes at the endpoints, and is positive at interior gridpoints; $p + \varepsilon b$ exceeds $B$ at the tie for every $\varepsilon > 0$, with unchanged endpoint values. ($\Rightarrow$) If all interior values are strictly below $B$, then the finitely many strict inequalities $q(j) < \max\{q(0), q(N)\}$ persist for all $q$ near $p$, so $p$ is interior. The tie bound: $B - p$ is a polynomial of degree $\leq n-1$ vanishing at each tie, so a nonconstant $p$ has at most $n-1$ of them. ☐

*Remark* 4.2. In the continuous theory [2, Theorem 6.2] the boundary has two mechanisms: interior tangency, and a critical endpoint at which the maximum is attained. On the grid the second collapses into the first: "critical at a dominant endpoint" means $\Delta p_0 = 0$ with $p(0) = B$, i.e. $p(1) = B$—already an interior tie. The discrete boundary is stratified purely by tie patterns.

*Remark* 4.3. The discrete stratification is *not* the restriction of the continuous one: a polynomial may dip below its endpoint maximum between gridpoints, or exceed it there, without the grid noticing. Consequently $\mathcal{M}(n-1, N)$ strictly contains the set of polynomials satisfying the continuous maximum principle on $[0, N]$, and the inclusion is strict already for cubics. The polygonal walls of Remark 2.4 live in this gap.

## 5. The maximal overshoot, exactly

Let $E(n,N)$ be the supremum of $\max_j x_j - \max\{x_0, x_N\}$ over all sequences with $\Delta^n x \geq 0$ and $|\Delta^i x_0| \leq 1$ for $1 \leq i \leq n-2$ (no constraint on $x_0$ or $x_N$), and let

$$P_{n,N}(j) \;=\; \sum_{i=1}^{n-2}\Big[\binom{j}{i} - \binom{N}{i}\frac{\binom{j}{n-1}}{\binom{N}{n-1}}\Big]$$

be the member of the class with data $(0,1,\ldots,1;\,0)$, vanishing at $j=0$ and $j=N$.

**Theorem 5.1.** *$E(n,N) = \max_{0\leq j\leq N} P_{n,N}(j)$, attained by $P_{n,N}$ itself. In particular*

$$E(3,N) \;=\; \frac{\lfloor N^2/4\rfloor}{N-1},$$

*and for fixed $n \geq 3$, as $N\to\infty$,*

$$E(n,N) \;=\; \frac{N^{\,n-2}}{(n-2)!}\max_{0\leq t\leq 1} t^{\,n-2}(1-t) \;+\; O\big(N^{\,n-3}\big) \;=\; \frac{N^{\,n-2}}{(n-2)!}\,\frac{(n-2)^{n-2}}{(n-1)^{n-1}} \;+\; O\big(N^{\,n-3}\big).$$

*Proof.* As in [2, Theorem 6.8], in two steps. *Step 1:* for $1 \leq i \leq n-2$,

$$\frac{\partial\, v_d(j)}{\partial c_i} \;=\; \binom{j}{i} - \binom{N}{i}\frac{\binom{j}{n-1}}{\binom{N}{n-1}} \;\geq\; 0,$$

because the ratio $\binom{j}{i}/\binom{N}{i} = \prod_{l=0}^{i-1}\frac{j-l}{N-l}$ is nonincreasing in $i$ for $0\leq j\leq N$ (each new factor lies in $[0,1]$; for $j<i$ both sides vanish). The benchmark does not involve $c_i$, so the supremum saturates $c_1 = \cdots = c_{n-2} = 1$. *Step 2:* subtracting $c_0$, the overshoot depends only on $t = \beta - c_0$ through $O(t) = \varphi(t) - t^+$ with $\varphi(t) = \max_j[\,\cdots + t\binom{j}{n-1}/\binom{N}{n-1}\,]$; since the coefficients $\binom{j}{n-1}/\binom{N}{n-1}$ lie in $[0,1]$, $\varphi$ is nondecreasing with $\varphi(t)-t$ nonincreasing, so $O(t) \leq O(0) = \max_j P_{n,N}(j)$ for all $t\in\mathbb{R}$, with equality at $t=0$.

For $n=3$: $P_{3,N}(j) = j - N\binom{j}{2}/\binom{N}{2} = j(N-j)/(N-1)$, maximized over integers at $j = \lfloor N/2\rfloor$, giving $\lfloor N/2\rfloor\lceil N/2\rceil/(N-1) = \lfloor N^2/4\rfloor/(N-1)$. For the asymptotics, put $j = tN$: $\binom{j}{i} = (tN)^i/i! + O(N^{i-1})$ uniformly, so $P_{n,N}(tN) = \frac{N^{n-2}}{(n-2)!}t^{n-2}(1-t) + O(N^{n-3})$, the top index $i = n-2$ dominating; maximizing $t^{n-2}(1-t)$ at $t = \frac{n-2}{n-1}$ gives the constant. □

*Remark* 5.2. For $n=3$ the extremal sequence is $j(N-j)/(N-1)$—the discrete incarnation of $x - x^2$, closing the same loop as in [2]: the counterexample that shows $n$-convexity alone does not force the maximum principle is also the extremal witness for how badly it fails. Unlike the continuous unit-interval normalization, where $E(n)\uparrow e-1$, here the top difference dominates and the overshoot grows polynomially in the length; the two regimes are reconciled by the scaling $\Delta^i \sim N^{-i}$.

## 6. Questions

**Question 6.1.** For $n=4$, Theorem 2.3(iv) presents $\mathcal{D}(4,N)$ as a union of two polyhedral cones, while [2, Theorem 5.6] describes the continuum limit, whose non-forcing region is bounded by the parabolic tangency wall. Determine the facet structure: which of the $N-1$ interior gridpoint inequalities, together with the endpoint-dominance inequality, are facets of $\mathcal{A},\mathcal{B}$; how the facet count grows with $N$; and in what sense (after the natural rescaling of the difference data) the polygonal walls converge to the parabola $16|A|c_1 = -(c_2^+)^2$ of the continuous description.

**Question 6.2.** Stratify $\partial\mathcal{M}(n-1,N)$ by tie patterns (which subsets of gridpoints can tie, with the bound of Theorem 4.1), determine which patterns are realized and how the strata glue, and compare with the contact-type poset of [2, Theorems 6.3–6.4], noting Remark 4.3.

**Question 6.3.** Unify with [3]: replace the $r$-th level of the difference tower by a weighted difference $(\Delta_{w^{(r)}} x)_j = (x_{j+1} - x_j)/w_j^{(r)}$ with positive, level-dependent weights—the discrete analogue of the Pólya factorization. The chain lemma is insensitive to the weights, so the core theory should persist; the order-three constant should become a weighted trapezoidal weight, interpolating between Theorem 3.1 and [3, Theorem 3.1].

## Acknowledgments

Third of a series: this note discretizes [2], whose methods were freed from polynomials in [3]; all three descend from [1].

## References

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