FORCING DIRICHLET DATA FOR POLYHARMONIC MAXIMUM PRINCIPLES

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Abstract. Sixth in a series descending from [5]: we determine which Dirichlet (clamped) data $u, \partial_\nu u, \ldots, \partial_\nu^{m-1} u$ on $\partial\Omega$ force the maximum principle for the class $\Delta^m u \geq 0$. The answer is a dichotomy that identifies the forcing program with positivity preservation. On a smooth bounded domain, the following are equivalent: the polyharmonic Green function with Dirichlet conditions is nonnegative; the class has a one-signed comparison with its polyharmonic interpolant (greatest element for m odd, least for m even); some Dirichlet data forces; every constant boundary value forces. And when the Green function changes sign, every nonempty class is pointwise unbounded in the forbidden direction, so *no* Dirichlet data forces—on sufficiently eccentric ellipses the clamped plate's forcing theory is empty, while on balls, by Boggio's theorem, it is complete. This answers Problem 7.3 of [10] and explains, in hindsight, why the first five papers of the series could avoid Green-function positivity: with Dirichlet data the avoidance is impossible. On the ball the constant-data theory is explicit and, unlike its Navier counterpart, *independent of the dimension*: the forcing cone is linearly isomorphic to the classical cone of polynomials of degree $\leq m-2$ nonnegative on an interval (so it is semidefinite-representable, and polyhedral exactly for $m \leq 3$); for the clamped plate ($m = 2$) the criterion is $\partial_\nu u \leq 0$; for $m = 3$ the cone is the sector $c_1 \geq 0$, $5c_1 \geq R\,c_2$, with sharp ratio $R/5$ attained at the center and lower ratio 0 as a boundary limit; an alternating criterion $(-1)^{m-k} c_k \geq 0$ forces at every order; and the worst-case overshoot under $|c_k| \leq 1$ is $(5R + R^2)/8$ for $m = 3$—equal to $3/4$ on the unit ball in every dimension. For the clamped plate the slope criterion extends to nonconstant slope data on every positivity-preserving domain, by differentiating the Green function twice at the boundary.

1. Introduction

The preceding papers of this series [6, 7, 8, 9, 10] answered, in successively wider settings, one question: for a class of functions defined by a differential inequality and prescribed data, which data *force* the maximum principle? A structural pattern emerged. In each setting the class had a pointwise-greatest (or least) element, the interpolant from the kernel of the operator, so forcing reduced to checking one function; and in each setting that one-signed comparison was obtained *without* any positivity theory for higher-order Green functions—by disconjugacy in one variable [6, 7, 8], by a degenerate radial tower on the ball [9], by a cascade of second-order problems under Navier conditions [10]. Problem 7.3 of [10] asked what happens when the avoidance stops being possible: Dirichlet data

$$u,\ \partial_\nu u,\ \ldots,\ \partial_\nu^{m-1} u \quad \text{on } \partial\Omega,$$

the clamped conditions, for which the comparison function is controlled by the polyharmonic Green function itself and by nothing softer.

This paper answers that problem. The answer (Theorem 3.1) is that the forcing program and positivity preservation are *the same phenomenon*: on a smooth bounded domain, the class

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$\Delta^m u \geq 0$ with Dirichlet data admits a one-signed comparison with its interpolant if and only if the clamped Green function $G_{m,\Omega}$ is nonnegative, if and only if some Dirichlet data forces the relevant principle, if and only if every constant boundary value does. When positivity fails, nothing survives: every nonempty data class is pointwise unbounded in the forbidden direction, so no data forces and no class has a one-signed barrier. Thus on the disc the clamped plate has a complete forcing theory, and on a sufficiently eccentric ellipse it has none—the forcing cone collapses from a nontrivial sector to the empty set as the domain crosses the positivity frontier. The proof is short: one direction is the Green representation plus an approximation argument; the other concentrates the load $\Delta^m u$ near a point where the kernel has the wrong sign. A separate energy identity shows the *parity* half of the story—that for m odd no data can force the minimum principle, and for m even none can force the maximum principle—holds on every domain, positivity or not, exactly as for Navier data [10].

On balls, Boggio's theorem [1] supplies positivity in every dimension and at every order, and the constant-data theory becomes completely explicit (Section 4). Three features are worth advertising.

- **The constants are dimension-free.** The clamped interpolant for constant data on $B_R \subset \mathbb{R}^d$ is a polynomial in $t = |x|^2$ determined by radial derivative conditions in which d never appears (Lemma 4.1). Every criterion and constant below is therefore the same in all dimensions—in sharp contrast to the Navier constants of [10], which decay like $1/d$. Boggio's input depends on d; the output does not.
- **The forcing cone is a classical object.** Dividing the interpolant by its forced boundary root gives a linear isomorphism of the constant-data forcing cone onto the cone of polynomials of degree $\leq m-2$ nonnegative on an interval (Theorem 4.5). Its extreme rays, boundary, and semidefinite representation are classical (Markov–Lukács); it is polyhedral exactly for $m \leq 3$. The center-data ball problem of [9] landed on the endpoint-maximum cone $\mathcal{M}_{m-1}$ of [6], a more elaborate object; clamped data lands on the simpler classical cone.
- **Order three is a two-facet sector, again.** For $m = 3$ the criterion is

$$c_1 \geq 0 \qquad \text{and} \qquad 5\,c_1 \geq R\,c_2$$

(Theorem 4.3): the sharp ratio $R/5$ is attained at the center, the lower ratio 0 only as a boundary limit—the exact clamped counterpart of the torsion-ratio sector of [10], with $\big(R/5,\ 0\big)$ in place of $\big((d+4)R^2/(4d(d+2)),\ R^2/(d(d+2))\big)$.

Section 4 also proves an alternating criterion at every order, with the same sign pattern $(-1)^{m-k}c_k \geq 0$ as the Navier criterion [10, Theorem 3.1], by a two-line sign induction on the operator $(2r)^{-1}\partial_r$. Section 5 extends the clamped-plate slope criterion to nonconstant slope data on arbitrary positivity-preserving domains: differentiating the Green function twice at its boundary double zero shows the slope kernel is nonnegative wherever the Green function is. Section 6 computes the exact worst-case overshoot for constant data; Section 7 records the one-dimensional meaning of all this (two-point Hermite data, split (m,m)—a data pattern not covered by [6, 7, 8]); Section 8 collects problems.

All numerical claims have been verified in exact rational arithmetic; the script is available with the source.

2. Setting

Fix $m \geq 1$, $\gamma \in (0,1)$, and a bounded domain $\Omega \subset \mathbb{R}^d$ with $\partial\Omega \in C^{2m,\gamma}$; call this hypothesis (S). For data $g = (g_0, \ldots, g_{m-1})$, $g_k \in C(\partial\Omega)$, let

$$\mathcal{K}_m(g) \;=\; \Big\{ u \in C^{2m}(\overline{\Omega}) \;:\; \Delta^m u \geq 0 \text{ in } \Omega, \quad \partial_\nu^k u = g_k \text{ on } \partial\Omega \ (0 \leq k \leq m-1) \Big\},$$

where ν is the outward unit normal. As in the earlier papers, g is *max-forcing* if every $u \in \mathcal{K}_m(g)$ satisfies $u \le \max_{\partial\Omega} g_0$ on $\overline{\Omega}$, and *min-forcing* if every member satisfies $u \ge \min_{\partial\Omega} g_0$. Forcing is only of interest when $\mathcal{K}_m(g) \neq \emptyset$ (an empty class forces vacuously); every negative statement below—"no data forces"—is to be read over data with nonempty class. (We work in the classical class $C^{2m}(\overline{\Omega})$ for cleanliness; we have not pursued minimal regularity.)

Call g *admissible* if $g_k \in C^{2m-k,\gamma}(\partial\Omega)$ for each k. Under (S), Schauder theory for the clamped problem [3, §2.5.1] then provides a unique *Dirichlet interpolant* $v_g \in C^{2m,\gamma}(\overline{\Omega})$ with $\Delta^m v_g = 0$ and data g (extend the data to some $\Phi \in C^{2m,\gamma}(\overline{\Omega})$ and solve the clamped problem with right-hand side $-\Delta^m\Phi$); in particular $v_g \in \mathcal{K}_m(g)$. Uniqueness is by the energy form: with

$$B[\varphi,\psi] \;=\; \begin{cases} \int_\Omega \Delta^\kappa\varphi\,\Delta^\kappa\psi, & m = 2\kappa,\\ \int_\Omega \nabla\Delta^\kappa\varphi\cdot\nabla\Delta^\kappa\psi, & m = 2\kappa+1, \end{cases}$$

B is an inner product on $H_0^m(\Omega)$ whose norm is equivalent to the H^m norm [3, §2.2], and a polyharmonic function with zero Dirichlet data lies in H_0^m and is annihilated by B. Throughout we use the standard fact that a $C^m(\overline{\Omega})$ function with zero Dirichlet data has *all* partial derivatives of order $\le m-1$ vanishing on $\partial\Omega$ (tangential derivatives of the vanishing traces vanish), hence lies in $H_0^m(\Omega)$.

Under (S) the clamped problem has a Green function $G = G_{m,\Omega}$: for $f \in C^{0,\gamma}(\overline{\Omega})$ the unique clamped solution of $(-\Delta)^m w = f$ is $w = \int_\Omega G(\cdot,y)f(y)\,dy$; G is symmetric and smooth off the diagonal [3, Ch. 2]. Following [3]:

Definition 2.1. Ω is *positivity preserving* (PP) for the order-m clamped problem if $G_{m,\Omega}(x,y) \ge 0$ for all $x \neq y$ in Ω.

Balls are PP in every dimension and at every order (Boggio [1]; see [3, Ch. 2]), as are small smooth perturbations of the disc for the clamped plate (see [3, Ch. 6] for the catalog of such results). PP fails, for $m = 2$, $d = 2$, on sufficiently eccentric ellipses (Garabedian); historically it also fails on long thin rectangles (Duffin) and on the square (Coffman–Duffin), but these domains have corners and lie outside hypothesis (S), so only the smooth examples are used below. See [3, Ch. 1] and the references therein.

Two lemmas do all the general work. The first converts PP into a one-signed comparison for the whole class, not just for Hölder right-hand sides.

Lemma 2.2 (Comparison)**.** *Assume (S) and PP. If $h \in C^{2m}(\overline{\Omega})$ has zero Dirichlet data and $\Delta^m h \ge 0$ in Ω, then $(-1)^m h \ge 0$ in Ω.*

Proof. Let $f = \Delta^m h \in C(\overline{\Omega})$, $f \ge 0$. Extend f continuously to $\mathbb{R}^d$, let φ_j be mollifications converging to f uniformly on $\overline{\Omega}$, and set $f_j := \varphi_j + \sup_{\overline{\Omega}}|\varphi_j - f|$: then $f_j \in C^\infty(\overline{\Omega})$, $f_j \ge f \ge 0$, and $f_j \to f$ uniformly on $\overline{\Omega}$. Let $h_j \in C^{2m,\gamma}(\overline{\Omega})$ be the clamped solution of $\Delta^m h_j = f_j$ [3, §2.5.1]. Since $(-\Delta)^m h_j = (-1)^m f_j$, the Green representation gives $(-1)^m h_j = \int_\Omega G(\cdot,y)f_j(y)\,dy \ge 0$ by PP.

Both h and h_j lie in $H_0^m(\Omega)$, and $w := h - h_j$ satisfies, for every $\varphi \in C_c^\infty(\Omega)$,

$$B[w,\varphi] \;=\; \int_\Omega (-\Delta)^m w\,\varphi \;=\; (-1)^m \int_\Omega (f - f_j)\,\varphi$$

(integration by parts, no boundary terms as φ has compact support); by density and continuity of both sides in $\|\varphi\|_{H^m}$, the identity extends to $\varphi \in H_0^m$. Taking $\varphi = w$ and using the coercivity of B,

$$c\,\|w\|_{H^m}^2 \;\le\; B[w,w] \;\le\; \|f - f_j\|_{L^2}\,\|w\|_{L^2},$$

so $h_j \to h$ in H_0^m, and a subsequence converges a.e. Hence $(-1)^m h \ge 0$ a.e., and everywhere by continuity. □

The second lemma is the parity no-go, and it needs no positivity at all: it is the clamped analogue of [10, Theorem 2.2], with the energy identity replacing the iterated torsion function.

Lemma 2.3 (Parity). *Assume (S) and let $\mathcal{K}_m(g) \neq \emptyset$. If m is odd, members of $\mathcal{K}_m(g)$ are unbounded below at some point of Ω, so no data is min-forcing; if m is even, they are unbounded above at some point, so no data is max-forcing.*

Proof. Pick $\zeta \in C_c^\infty(\Omega)$, $\zeta \geq 0$, $\zeta \not\equiv 0$, and let w solve the clamped problem $\Delta^m w = \zeta$. Then $w \in H_0^m$, $w \not\equiv 0$, and

$$0 \;<\; B[w,w] \;=\; \int_\Omega (-\Delta)^m w\, w \;=\; (-1)^m \int_\Omega \zeta\, w,$$

so $(-1)^m w > 0$ somewhere in Ω. If $u_0 \in \mathcal{K}_m(g)$, then $u_0 + t\,w \in \mathcal{K}_m(g)$ for all $t \geq 0$ (the data are untouched, and Δ^m only gains $t\zeta \geq 0$), and as $t \to \infty$ this family is unbounded below at a point where $w < 0$ (m odd: $\int \zeta w < 0$ forces such a point), respectively unbounded above where $w > 0$ (m even). □

3. The dichotomy

Theorem 3.1 (Dichotomy). *Assume (S) with m odd. The following are equivalent.*

(a) *Ω is PP: $G_{m,\Omega} \geq 0$ off the diagonal.*
(b) *Every $h \in C^{2m}(\overline{\Omega})$ with zero Dirichlet data and $\Delta^m h \geq 0$ satisfies $h \leq 0$.*
(c) *For every admissible g, the interpolant v_g is the pointwise-greatest element of $\mathcal{K}_m(g)$.*
(d) *Some admissible g is max-forcing.*
(e) *Every constant boundary value—the data $(c_0, 0, \ldots, 0)$—is max-forcing.*

Under PP, an admissible g is max-forcing if and only if $v_g \leq \max_{\partial\Omega} g_0$ on Ω. If PP fails, then every nonempty class $\mathcal{K}_m(g)$ is unbounded above at some interior point (one point serves all classes); consequently no data (admissible or not) is max-forcing, and no nonempty class has a pointwise-greatest element.

For m even the same equivalences and statements hold with "greatest, max-forcing, $h \leq 0$, unbounded above" replaced by "least, min-forcing, $h \geq 0$, unbounded below".

Proof. (a) ⇒ (b) is Lemma 2.2. (b) ⇒ (c): apply (b) to $h = u - v_g$ for $u \in \mathcal{K}_m(g)$. (c) ⇒ (e): the interpolant of $(c_0, 0, \ldots, 0)$ is the constant c_0, so every member is $\leq c_0 = \max_{\partial\Omega} g_0$. (e) ⇒ (d) is trivial.

(d) ⇒ (a) by contraposition; this is also the failure statement. Suppose $G(x_0, y_0) < 0$ for some $x_0 \neq y_0$. Since G is continuous off the diagonal, there are $\delta, r > 0$ with $\overline{B_r(y_0)} \subset \Omega \setminus \{x_0\}$ and $G(x_0, \cdot) \leq -\delta$ on $B_r(y_0)$. Take $f \in C_c^\infty(B_r(y_0))$, $f \geq 0$, $\int f = 1$, and let w be the clamped solution of $\Delta^m w = f$; the Green representation gives

$$w(x_0) \;=\; (-1)^m \int_\Omega G(x_0, y) f(y)\, dy \;\geq\; \delta \;>\; 0 \qquad (m \text{ odd}).$$

If $\mathcal{K}_m(g) \neq \emptyset$, pick u_0 in it: $u_0 + t\,w \in \mathcal{K}_m(g)$ for $t \geq 0$ and is unbounded above at x_0. So no g with a nonempty class is max-forcing, and a pointwise-greatest element w^* would give the contradiction $u_0(x_0) + t\,w(x_0) \leq w^*(x_0) < \infty$. ($m$ even: $w(x_0) \leq -\delta$, unbounded below, blocking min-forcing and least elements.)

The barrier criterion under PP: if $v_g \leq \max g_0$, then by (c) every member is $\leq v_g \leq \max g_0$; conversely v_g is itself a member. The m-even mirror follows by the same proofs with the signs reversed; note Lemma 2.3 already rules out the opposite principle in each parity, on every domain. □

Corollary 3.2. *For the clamped plate ($m = 2$, $d = 2$): on the disc, and on every smooth positivity-preserving perturbation of it* [3, Ch. 6]*, the min-forcing theory is nontrivial and governed by the interpolant barrier; on sufficiently eccentric Garabedian ellipses* [3, Ch. 1]*, no Dirichlet data whatsoever is min-forcing. (The other classical sign-change examples—Duffin's long rectangles and the Coffman–Duffin square—have corners and lie outside (S); extending the dichotomy to polygonal domains would require a weak-solution framework we do not develop here.) As a smooth family of domains crosses the positivity frontier, the forcing set collapses from a nonempty interpolant-defined set—explicit on the disc by Theorem 4.5—to $\emptyset$.*

Remark 3.3. The dichotomy explains the architecture of this series. Papers [6, 7, 8, 9, 10] all obtained greatest elements from disconjugacy-type structures—one-dimensional sign chains, radial towers, second-order cascades—and therefore never met a Green-function sign condition; [7, §5] even reached radial data on *annuli*, where PP for the clamped plate can fail, because radial disconjugacy sidesteps the clamped Green function entirely. Theorem 3.1 shows this was not good fortune but the exact boundary of the soft theory: for genuinely clamped data the forcing program lives and dies with positivity preservation, on Boggio's territory.

4. Balls: the explicit constant-data theory

Let $B_R \subset \mathbb{R}^d$ ($d \geq 1$; for $d = 1$ an interval, with "radial" meaning even) and prescribe *constant* data $c = (c_0, c_1, \ldots, c_{m-1})$, $\partial_\nu^k u = c_k$. By Boggio's theorem B_R is PP for every m and d [1, 3] (for $d = 1$ the required sign of the two-point (m, m) Green function is classical disconjugacy theory [2]), so Theorem 3.1 applies: for m odd, c is max-forcing iff $v_c \leq c_0$, and for m even, min-forcing iff $v_c \geq c_0$. Write $t = |x|^2$, $T = R^2$.

Lemma 4.1 (The interpolant, dimension-free)**.** *The system*

$$v \;=\; \sum_{j=0}^{m-1} a_j\, |x|^{2j}, \qquad \frac{d^k}{dr^k}\Big(\sum_j a_j r^{2j}\Big)\Big|_{r=R} = c_k \quad (0 \leq k \leq m-1)$$

has a unique solution, and v is the Dirichlet interpolant v_c on $B_R \subset \mathbb{R}^d$ for every d. The coefficient system does not involve d; consequently $\psi_c := v_c - c_0$, as a polynomial in t with $\psi_c(T) = 0$, and every constant-data criterion below, are independent of the dimension.

Proof. Each $|x|^{2j}$ with $j \leq m-1$ is polyharmonic of order m in every $\mathbb{R}^d$: the radial Laplacian acts on polynomials in t by $\Delta t^j = 2j(2j + d - 2)\, t^{j-1}$, lowering the degree by one, so Δ^m annihilates degrees $\leq m-1$. The $m \times m$ system is nonsingular: if all r-derivatives of order $\leq m-1$ of the even polynomial $p(r) = \sum a_j r^{2j}$ (degree $\leq 2m-2$) vanish at $r = R$, then p vanishes to order m at R, hence by evenness at $-R$, so $(r^2 - R^2)^m$ divides p; the degree forces $p \equiv 0$. The resulting v is polyharmonic with the prescribed data, hence equals v_c by uniqueness of the clamped problem. Dimension-freeness is visible: the system involves only r-derivatives at R. □

Remark 4.2. Under the dilation $u \mapsto u(R\,\cdot)$ the data transform by $c_k \mapsto R^k c_k$, so one may normalize $R = 1$; we keep R explicit in the headline formulas. Contrast the dimension-freeness with [10]: the Navier data $\Delta^k u$ see the operator, and the torsion-ratio constants decay like $1/d$; the clamped data $\partial_\nu^k u$ see only the radial profile, and the profile space $\{1, r^2, \ldots, r^{2m-2}\}$ is the same in every dimension. Boggio's input is d-dependent; the output is not.

Theorem 4.3 (Order three: the sector)**.** *Let $m = 3$, any $d \geq 1$, any $R > 0$. The interpolant is $v = c_0 + c_1 A + c_2 B$ with*

$$A \;=\; \frac{(t-T)(5R^2 - t)}{8R^3} \;\leq\; 0, \qquad B \;=\; \frac{(t-T)^2}{8R^2} \;\geq\; 0 \qquad (t = |x|^2,\ T = R^2),$$

and the constant data (c_0, c_1, c_2) *is max-forcing if and only if*

$$c_1 \;\geq\; 0 \qquad \text{and} \qquad 5\,c_1 \;\geq\; R\,c_2.$$

Equivalently, with $\sigma := B/(-A) = R\,(T-t)/(5R^2-t)$ *on* B_R*,*

$$c_1 \;\geq\; \sigma_{\max}\,c_2 \;\; (c_2 \geq 0), \qquad c_1 \;\geq\; \sigma_{\min}\,c_2 \;\; (c_2 \leq 0),$$

where $\sigma_{\max} = R/5$ *is attained at the center and* $\sigma_{\min} = 0$ *is a boundary limit, not attained.*

Proof. That A and B solve the two cardinal systems of Lemma 4.1 is a direct check (A: value 0, $\partial_\nu = 1$, $\partial_\nu^2 = 0$ at $r = R$; B: value 0, $\partial_\nu = 0$, $\partial_\nu^2 = 1$). By Theorem 3.1, forcing means $\psi := c_1 A + c_2 B \leq 0$ on $[0,T]$. Since $\psi(T) = 0$, write $\psi = (t-T)\chi$ with χ linear; then $\psi \leq 0$ on $[0,T]$ iff $\chi \geq 0$ there, iff $\chi(0) \geq 0$ and $\chi(T) \geq 0$. Computing, $\chi(0) = (5c_1 - Rc_2)/(8R)$ and $\chi(T) = c_1/(2R)$, which is the sector. The ratio form follows by dividing $\psi \leq 0$ by $-A > 0$ on $(0,T)$; σ is strictly decreasing (numerator decreasing, denominator increasing), so its supremum $R/5$ sits at $t = 0$ and its infimum 0 is approached only as $t \to T$. □

Remark 4.4. The vanishing of $\sigma_{\min}$ is structural, not accidental: B has a *double* root at the boundary (it is the top cardinal interpolant; see Remark 4.6), while A has a simple one, so the ratio $B/(-A)$ dies linearly at $\partial\Omega$. The same boundary asymptotics ($A \sim -s$, $B \sim s^2/2$ in the boundary distance s) will force $\sigma_{\min} = 0$ on *any* PP domain on which the ratio description is valid; the nontrivial domain functional is $\sigma_{\max}$ alone. Compare the Navier sector [10, Theorem 4.2], where both torsion ratios are nontrivial: there the lower slope datum is an iterated *Laplacian*, here an iterated normal derivative, and the extra boundary vanishing is exactly one order.

Theorem 4.5 (All orders: the nonnegative-polynomial cone)**.** *Fix* $m \geq 2$*,* $R > 0$*, any* d*. The map* $c \mapsto \chi_c := \psi_c/(t-T)$ *is a linear isomorphism from* $\{(c_1, \ldots, c_{m-1})\} \cong \mathbb{R}^{m-1}$ *onto the polynomials of degree* $\leq m-2$*, and:*

(i) *for* m *odd,* c *is max-forcing iff* $\chi_c \geq 0$ *on* $[0,T]$*; for* m *even,* c *is min-forcing iff* $\chi_c \leq 0$ *on* $[0,T]$*;*
(ii) *the constant-data forcing cone is therefore linearly isomorphic to the classical cone* $\mathcal{P}_{m-2}$ *of polynomials of degree* $\leq m-2$ *nonnegative on an interval: it is closed, convex, semidefinite-representable via the Markov–Lukács theorem* [11, §1.21] *(as already exploited in* [6, §2]*), its extreme rays and boundary structure are classical* [4]*, and it is polyhedral exactly for* $m \leq 3$*;*
(iii) *in particular, for* $m = 2$*: min-forcing* $\iff$ $c_1 \leq 0$ *(the clamped plate stays above its edge height iff the prescribed outward edge slope is nonpositive).*

Proof. $\psi_c(T) = 0$ always, so χ_c is a polynomial of degree $\leq m-2$, depending linearly on c; injectivity is Lemma 4.1 (if $\chi_c = 0$ then $\psi_c = 0$, so all data vanish), and surjectivity follows from equal dimensions. On $[0,T]$, $\psi_c = (t-T)\chi_c \leq 0$ iff $\chi_c \geq 0$ on $[0,T)$, iff on $[0,T]$ by continuity; Theorem 3.1 converts this into forcing. The rescaling $t \mapsto Tt$ identifies $\mathcal{P}_{m-2}[0,T]$ with $\mathcal{P}_{m-2}[0,1]$. Nonnegative polynomials of degree ≤ 1 on an interval form a two-facet polyhedral sector (this is Theorem 4.3); degree 0 gives the half-line (iii). For degree ≥ 2, the section of the cone by the subspace of quadratics is the cone of quadratics nonnegative on $[0,1]$, whose extreme rays include the continuum $(t-s)^2$, $s \in [0,1]$ (see [4]); a polyhedral cone has polyhedral linear sections with finitely many extreme rays, so $\mathcal{P}_{m-2}$ is not polyhedral for any $m \geq 4$. □

Remark 4.6 (Cardinal interpolants). Write ψ_k for the cardinal interpolant ($c_j = \delta_{jk}$). Then $(t-T)^k$ divides ψ_k—the data force vanishing to order k at $r = R$, and evenness doubles it at

$-R$—and the top cardinal is exactly

$$\psi_{m-1} \;=\; \frac{(t-T)^{m-1}}{(m-1)!\,(2R)^{m-1}},$$

one-signed on $[0,T]$ with sign $(-1)^{m-1}$. Consequently the datum c_{m-1} alone (all other $c_k = 0$) forces—max for m odd, min for m even—if and only if $c_{m-1} \le 0$, for *both* parities: the sign of ψ_{m-1} flips in step with the principle being forced (m odd needs $c_{m-1}\psi_{m-1} \le 0$ with $\psi_{m-1} \ge 0$; m even needs $c_{m-1}\psi_{m-1} \ge 0$ with $\psi_{m-1} \le 0$), consistently with Theorem 4.5(iii) and Theorem 4.7. For $m = 4$, $R = 1$ the three cardinals are

$$\psi_1 = \tfrac{1}{16}(t-1)(t^2-4t+11), \qquad \psi_2 = -\tfrac{1}{16}(t-1)^2(t-3), \qquad \psi_3 = \tfrac{1}{48}(t-1)^3,$$

with alternating signs ≤ 0, ≥ 0, ≤ 0 on $[0,1]$.

Theorem 4.7 (Alternating criterion)**.** *On $B_R \subset \mathbb{R}^d$ (any d, any $m \ge 2$), constant data with*

$$(-1)^{m-k}\, c_k \;\ge\; 0 \qquad (1 \le k \le m-1)$$

is max-forcing for every c_0 when m is odd, and min-forcing for every c_0 when m is even—the same sign pattern as the Navier criterion [10, Theorem 3.1].

Proof. Let $\theta := (2r)^{-1}\partial_r$, so that $\frac{d}{dt} = \theta$ under $t = r^2$ and $\psi^{(k)}(T) = (\theta^k v)(R)$ for $k \ge 1$. We claim

$$\theta^k \;=\; \sum_{j=1}^{k} b_{kj}\, r^{\,j-2k}\, \partial_r^j \qquad \text{with} \qquad (-1)^{k-j}\, b_{kj} \;>\; 0.$$

Induction: θ^1 has $b_{11} = \frac{1}{2}$. Applying θ to a term $b\, r^a \partial^j$ with $a = j - 2k \le -k < 0$ produces $\frac{ab}{2}\, r^{a-2}\partial^j + \frac{b}{2}\, r^{a-1}\partial^{j+1}$: the first contributes to $b_{k+1,j}$ with the sign of $-b$, i.e. $(-1)^{(k+1)-j}$, and the second to $b_{k+1,j+1}$ with the sign of b, i.e. $(-1)^{(k+1)-(j+1)}$; all contributions to a fixed coefficient share one sign, so no cancellation occurs.

Now

$$\psi^{(k)}(T) \;=\; \sum_{j=1}^{k} b_{kj}\, R^{\,j-2k}\, c_j,$$

and under the hypothesis each summand has sign $(-1)^{k-j}(-1)^{m-j} = (-1)^{k+m}$; hence we obtain $(-1)^{k+m}\psi^{(k)}(T) \ge 0$ for all $k \ge 1$. Taylor expansion at T (exact, ψ is a polynomial) gives, for $t \in [0,T]$,

$$\psi(t) \;=\; \sum_{k\ge1} \frac{\psi^{(k)}(T)}{k!}\,(t-T)^k \;=\; \sum_{k\ge1} (-1)^k\, \frac{\psi^{(k)}(T)}{k!}\,(T-t)^k,$$

in which every term has sign $(-1)^k(-1)^{k+m} = (-1)^m$. So $\psi \le 0$ throughout for m odd and $\psi \ge 0$ for m even, and Theorem 3.1 concludes. □

Remark 4.8. For $m = 3$ the alternating orthant $c_1 \ge 0 \ge c_2$ sits strictly inside the sector of Theorem 4.3, which tolerates c_2 up to $5c_1/R$. Unlike the Navier criterion, which the cascade delivers on every bounded domain, the proof above is genuinely radial; the general-PP-domain version is Problem 8.1 (for $m = 2$ it is Proposition 5.1 below).

5. The clamped plate on general PP domains

For $m = 2$ the alternating criterion has a full nonconstant-data extension on every PP domain. The mechanism deserves display: positivity of the Green function *differentiates* to positivity of the slope kernel at the boundary, because a nonnegative function with a double zero has a nonnegative second derivative.

Proposition 5.1 (Slope criterion for the clamped plate)**.** *Assume (S) with $m = 2$ and let Ω be PP. Let $g = (g_0, g_1)$ be admissible with $g_0 \equiv c_0$ constant and $g_1 \le 0$ on $\partial\Omega$. Then every $u \in \mathcal{K}_2(g)$ satisfies $u \ge c_0$ on $\overline{\Omega}$; that is, g is min-forcing. On the ball, for constant $g_1 = c_1$, the converse holds: min-forcing iff $c_1 \le 0$.*

Proof. Fix $x \in \Omega$ and write $K(x, z) := \Delta_z G(x, z)$ for $z \in \partial\Omega$.

Step 1: $K \ge 0$. The function $G(x, \cdot)$ is smooth up to $\partial\Omega$ away from x [3, Prop. 4.17], nonnegative by PP, and has zero Dirichlet data, so its full gradient vanishes on $\partial\Omega$ (tangential derivatives of the zero trace vanish, and $\partial_\nu G = 0$ is the clamped condition). For $z \in \partial\Omega$ and the inward normal $-\nu$, Taylor expansion gives $0 \le G(x, z - s\nu) = \frac{s^2}{2}\,\partial^2_{\nu\nu} G(x, z) + o(s^2)$, so $\partial^2_{\nu\nu} G(x, z) \ge 0$. At a boundary point the Laplacian splits as $\Delta = \partial^2_{\nu\nu} + H\,\partial_\nu + \Delta_{\partial\Omega}$, where H is the mean curvature term and $\Delta_{\partial\Omega}$ the Laplace–Beltrami operator applied to the trace; the last two terms vanish because the trace and the normal derivative do, so $\Delta_z G(x, z) = \partial^2_{\nu\nu} G(x, z) \ge 0$.

Step 2: representation. For $f \in C_c^\infty(\Omega)$ let $w_f = \int_\Omega G(\cdot, y) f(y)\,dy$ be the clamped solution of $\Delta^2 w_f = f$. Green's identity twice, for the biharmonic interpolant $v = v_g$ and w_f (whose value and normal derivative vanish on $\partial\Omega$):

$$\int_\Omega v\,f \;=\; \int_\Omega v\,\Delta^2 w_f \;=\; \int_{\partial\Omega} \big[\, g_0\,\partial_\nu \Delta w_f \;-\; g_1\,\Delta w_f \,\big]\, dS.$$

Differentiating under the integral sign (legitimate: $G(y, \cdot)$ is smooth up to $\partial\Omega$, uniformly for y in the compact support of f, by [3, Prop. 4.17]) and using the symmetry of G, $\Delta w_f(z) = \int K(y, z) f(y)\,dy$ and similarly for $\partial_\nu \Delta w_f$; Fubini then gives, since f was arbitrary,

$$v(x) \;=\; \int_{\partial\Omega} \Big[\, g_0(z)\,\partial_\nu \Delta_z G(x, z) \;-\; g_1(z)\,K(x, z) \Big]\, dS(z) \qquad (x \in \Omega).$$

Applying this to $v \equiv 1$ (data $(1, 0)$) shows $\int_{\partial\Omega} \partial_\nu \Delta_z G(x, z)\, dS(z) = 1$.

Step 3. For $g_0 \equiv c_0$ and $g_1 \le 0$: $v_g(x) = c_0 - \int K(x, z)\, g_1(z)\, dS \ge c_0$. By Theorem 3.1 ($m$ even, PP), every member satisfies $u \ge v_g \ge c_0$. The ball converse is Theorem 4.5(iii). ☐

Remark 5.2. The one-dimensional instance is checked exactly in the verification script: on an interval, the two boundary kernels $\partial^2 G/\partial z^2$ at the endpoints are nonnegative and reproduce the clamped cubic interpolant. Extending Proposition 5.1 to nonconstant g_0 requires positivity of the *other* kernel $\partial_\nu \Delta_z G$, and to $m \ge 3$ positivity of the intermediate kernels; see Problem 8.1.

6. The overshoot

Theorem 6.1 (Overshoot)**.** *Assume (S) and PP, with m odd, and let A_k denote the interpolant of the constant data e_k ($c_j = \delta_{jk}$). Over all constant data with $|c_k| \le 1$ for $1 \le k \le m - 1$ and c_0 arbitrary, and all members,*

$$E_m(\Omega) \;:=\; \sup\Big(\max_{\overline{\Omega}} u - c_0\Big) \;=\; \max_{\overline{\Omega}} \;\sum_{k=1}^{m-1} |A_k|,$$

attained by an interpolant with $c_k \in \{\pm 1\}$*. For* m *even the mirror statement bounds* $c_0 - \min u$*. On the ball (*$t = |x|^2$*,* $T = R^2$*, any* d*):*

$$E_3(B_R) \;=\; \max_{[0,T]}\bigl(-A + B\bigr) \;=\; \frac{5R + R^2}{8} \quad \textit{at the center;} \qquad E_2(B_R) \;=\; \frac{R}{2}.$$

In particular $E_3(B_1) = 3/4$ *in every dimension. (The mixed powers of* R *reflect the normalizations* $|\partial_\nu u| \le 1$*,* $|\partial_\nu^2 u| \le 1$*: the two data scale differently.)*

Proof. By Theorem 3.1, $u \le v_c = c_0 + \sum_{k\ge1} c_k A_k \le c_0 + \sum_{k\ge1} |A_k|$ pointwise. For attainment, choose x^* maximizing $\sum |A_k|$, set $c_k := \operatorname{sign} A_k(x^*)$ (with $+1$ at zeros) and $c_0 := 0$: the interpolant is a member of its own class and reaches $\sum |A_k(x^*)|$ at x^*. On the ball with $m = 3$: $A_1 = A \le 0$ and $A_2 = B \ge 0$ from Theorem 4.3, and both $-A = (T-t)(5R^2 - t)/(8R^3)$ and $B = (T-t)^2/(8R^2)$ are products of positive decreasing factors on $[0, T]$, so the maximum of $-A + B$ is at $t = 0$: $5R/8 + R^2/8$. For $m = 2$, $|A_1| = (T-t)/(2R)$ peaks at the center with value $R/2$. □

7. One dimension: two-point Hermite data

For $d = 1$, $\Omega = (a, b)$, the clamped class is $u^{(2m)} \ge 0$ with prescribed $u^{(k)}(a)$ and $u^{(k)}(b)$ for $0 \le k \le m-1$: two-point Hermite data with the balanced split (m, m). This pattern is covered by none of the earlier papers: [6, 7, 8] prescribe $n-1$ conditions at one endpoint and one at the other, and [10] prescribes even-order derivatives at both ends (the Lidstone pattern). The interval is PP at every order—classical disconjugacy theory gives the sign $(-1)^m$ of the (m, m) two-point Green function of $u^{(2m)}$ [2]—so Theorem 3.1 applies verbatim: the class lies below (for m odd) or above (for m even) its two-point Hermite interpolant, and forcing is decided by that single polynomial. The results of Section 4 describe the *symmetric* slice (data even under reflection); the full asymmetric (m, m) forcing cone in $\mathbb{R}^{2m}$ is Problem 8.3. Even the balanced parity statement—$u^{(2m)} \ge 0$ with $u, u', \ldots, u^{(m-1)}$ pinned at both ends forces a one-signed deviation from the Hermite polynomial, with sign $(-1)^m$—though certainly known in the interpolation literature, plays a role here (it is the $d = 1$ Boggio input) that we have not seen exploited for maximum principles.

8. Problems

Problem 8.1. Nonconstant data. On the ball, extend Section 4 to nonconstant g_k: by the cascade-free representation of Section 5 this is exactly the question of the signs of the m boundary kernels of the clamped problem (the polyharmonic "Poisson kernels"). Positivity of the slope kernel on PP domains is Proposition 5.1, and the same double-zero argument signs the *top* kernel for every m; the intermediate kernels, and the g_0-kernel $\partial_\nu \Delta_z G$, are open here even on the disc, though we expect ball formulas to be classical for some pairs (m, d).

Problem 8.2. The clamped ratio as a domain functional. For $m = 3$ constant data on a general PP domain, is the forcing cone always the sector $\{c_1 \ge \sigma_{\max}(\Omega)\, c_2^+\}$, with $\sigma_{\max}(\Omega) = \sup_\Omega B_\Omega/(-A_\Omega)$ for the two cardinal interpolants? (This requires $A_\Omega \le 0 \le B_\Omega$, an instance of Problem 8.1; Remark 4.4 predicts $\sigma_{\min} = 0$ always.) If so, develop the Saint-Venant theory of $\sigma_{\max}$—scale-covariance $\sigma_{\max}(\lambda\Omega) = \lambda\, \sigma_{\max}(\Omega)$ suggests comparison with the inradius—in parallel with the torsion ratios of [10, Problem 7.2]. Note the domain class itself is now constrained: the functional only exists on PP domains.

Problem 8.3. The asymmetric interval. Determine the full (m, m) two-point forcing cone on an interval (Section 7), starting with $m = 2$: which quadruples $(u(a), u'(a), u(b), u'(b))$ force

$u \geq \min\{u(a), u(b)\}$ for $u^{(4)} \geq 0$? The symmetric answer is $c_1 \leq 0$; the asymmetric cone lives in $\mathbb{R}^4$ and its geometry (polyhedral or not) is unknown.

Problem 8.4. Quantitative dichotomy. On domains where PP *just* fails, the negative part of the biharmonic Green function is small in various senses (see [3, Ch. 6]). Does a relaxed principle survive: $u \leq \max_{\partial\Omega} g_0 + C(\Omega)\,\varepsilon(g)$ with $C(\Omega)$ controlled by the negative part of G, interpolating between the full forcing theory on PP domains and its collapse in Theorem 3.1?

Acknowledgments

Sixth in a series descending from [5], after [6, 7, 8, 9, 10]; it answers Problem 7.3 of [10]. All exact constants were verified in rational arithmetic.

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