WHICH DATA FORCE THE MAXIMUM PRINCIPLE FOR RADIAL POLYHARMONIC FUNCTIONS ON A BALL?

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Abstract. Maximum principles famously fail for polyharmonic operators. This note carries the forcing framework of [3, 4] to the simplest genuinely polyharmonic setting: radial functions $u \in C^{2m}(\overline{B}_R)$ with $\Delta^m u \geq 0$ on a ball in $\mathbb{R}^d$, with prescribed center data $\Delta^k u(0)$ $(0 \leq k \leq m-2)$ and boundary value $u|_{\partial B_R}$. In the variable $t = |x|^2$ the radial Laplacian factors as a degenerate Pólya tower whose odd quasi-derivatives vanish automatically at the center for regular functions; the sign-chain argument then shows that the class has a pointwise-greatest element, the radial m-polyharmonic interpolant $\sum_{i<m} a_i |x|^{2i}$. Consequently the forcing problem is decided by one explicit function, and the forcing set is linearly isomorphic to the polynomial endpoint-maximum cone $\mathcal{M}_{m-1}$ of [3], so the entire geometry developed there—the complete description at $m = 4$, non-polyhedrality for $m \geq 4$, the boundary poset—transfers to the ball wholesale. Concretely: every radial u with $\Delta^2 u \geq 0$ attains its maximum at the center or the boundary, so for the biharmonic class all data force; and for the triharmonic class the sharp criterion is

$$\Delta u(0) \;\leq\; \frac{4d}{R^2}\left(u|_{\partial B_R} - u(0)\right)^+,$$

the constant 2 of [2, 3] reborn with its dimensional prefactor. The worst-case interior overshoot under $|\Delta^k u(0)| \leq 1$ is computed exactly; for $m = 3$ it is $R^2/(8d)$, attained by $\frac{1}{2d}\left(|x|^2 - |x|^4/R^2\right)$, the radial reincarnation of $x - x^2$. None of this uses Boggio's theorem: disconjugacy of the radial operator substitutes for Green-function positivity.

1. Introduction

For the Laplacian, the maximum principle is unconditional. For its powers it is famously not: Δ^m with $m \geq 2$ does not in general preserve positivity, Green functions of the clamped plate can change sign on general domains, and the positive theory that survives—Boggio's theorem and its descendants—lives on balls and their perturbations (see the monograph [1]). The series [2, 3, 4, 5] developed, in one variable, a complementary point of view: rather than asking *whether* the maximum principle holds for a class $\{Lf \geq 0\}$, ask *which prescribed data force it*, and answer by exhibiting a pointwise-greatest element of the data class—a barrier—which reduces the infinite-dimensional forcing problem to a single explicit function.

This note brings that program to polyharmonic operators in the setting where everything can be carried out completely: radial functions on a ball. Fix $d \geq 1$, $R > 0$, $m \geq 2$, and let $\overline{B}_R \subset \mathbb{R}^d$ be the closed ball. Consider

$$\mathcal{K}(\delta) \;=\; \Big\{u \in C^{2m}(\overline{B}_R) \text{ radial } : \; \Delta^m u \geq 0, \;\; \Delta^k u(0) = \gamma_k \; (0 \leq k \leq m-2), \;\; u|_{\partial B_R} = \beta\Big\},$$

where $\delta = (\gamma_0, \ldots, \gamma_{m-2}; \beta) \in \mathbb{R}^m$ is the prescribed *center-boundary data*. (We reserve the adjective *polyharmonic* for the equality class $\Delta^m u = 0$; the class above is its inequality companion, whose extremal members will turn out to be polyharmonic.) Call δ *forcing* if every $u \in \mathcal{K}(\delta)$

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satisfies the maximum principle

$$u \;\le\; \max\bigl\{u(0),\, \beta\bigr\} \quad \text{on } \overline{B}_R,$$

and write $\mathcal{D}_m(R,d) \subseteq \mathbb{R}^m$ for the set of forcing data. (For $m=1$, subharmonic functions, every data vector is forcing; the question begins at $m=2$.)

Our results parallel [3] exactly, and the parallel is the point:

- **Reduction** (Theorem 3.1). $\mathcal{K}(\delta)$ has a pointwise-greatest element: the unique radial m-polyharmonic function $v_\delta = \sum_{i=0}^{m-1} a_i |x|^{2i}$ with data δ. Hence δ is forcing iff v_δ itself satisfies the maximum principle, the forcing set is a union of two convex cones, and the *descending criterion* holds: $\Delta^k u(0) \le 0$ for $1 \le k \le m-2$ forces, for every boundary value.
- **Biharmonic** (Corollary 3.2). Every radial $u \in C^4(\overline{B}_R)$ with $\Delta^2 u \ge 0$ attains its maximum over the ball at the center or on the boundary; in particular $\mathcal{D}_2(R,d) = \mathbb{R}^2$.
- **Triharmonic, sharply** (Theorem 4.1). $\delta \in \mathcal{D}_3(R,d)$ if and only if

$$\Delta u(0) \;\le\; \frac{4d}{R^2}\bigl(\beta - u(0)\bigr)^+.$$

- **Inherited geometry** (Corollary 5.1). In suitable linear coordinates $\mathcal{D}_m(R,d)$ *is* the cone $\mathcal{M}_{m-1}$ of polynomials of degree $\le m-1$ with endpoint maximum studied in [3]: the complete description at $m=4$, the failure of polyhedrality for $m \ge 4$, and the boundary stratification all transfer verbatim.
- **Overshoot** (Theorem 5.2). Under $|\Delta^k u(0)| \le 1$ $(1 \le k \le m-2)$, the worst interior excess over $\max\{u(0),\beta\}$ is computed exactly; for $m=3$ it equals $R^2/(8d)$, attained by $\frac{1}{2d}(|x|^2 - |x|^4/R^2)$.

The engine is the change of variables $t = |x|^2$, under which the radial Laplacian becomes the degenerate operator $L = 4t\,\partial_t^2 + 2d\,\partial_t$ with Pólya-type factorization $LU = 4t^{1-\nu}(t^\nu U')'$, $\nu = d/2$. The weights are singular at the center, but regularity comes to the rescue: the odd quasi-derivatives $t^\nu(\Delta^k u)'$ of a regular radial function vanish at $t=0$ automatically, supplying exactly the data the sign chain of [4] needs at a boundary point where none is prescribed. Notably, Boggio's positivity theorem is never invoked: one-dimensional disconjugacy, surviving the degeneracy, does all the work—which is also why the method is indifferent to the sign pathologies of polyharmonic Green functions. For $d=1$ the statements specialize to even functions on a symmetric interval.

2. The degenerate tower at the center

Throughout, $\nu = d/2$, $T = R^2$, and for a radial u we write $U(t) = u(x)|_{|x|^2=t}$, $t \in [0,T]$. A computation gives

$$\Delta u \;=\; 4t\,U'' + 2d\,U' \;=\; 4\,t^{1-\nu}\bigl(t^\nu\,U'\bigr)' \;=:\; LU. \tag{1}$$

Lemma 2.1 (Regularity at the center). *If $w \in C^2(\overline{B}_R)$ is radial, then $W(t) = w|_{|x|^2=t}$ is $C^1[0,T]$, with $2d\,W'(0) = \Delta w(0)$; in particular $t^\nu W'(t) \to 0$ as $t \to 0^+$.*

Proof. Write $w = \tilde{w}(r)$, $r = |x|$. Differentiability of w at the origin forces $\tilde{w}'(0) = 0$, so by Taylor $\tilde{w}'(r) = \tilde{w}''(0)\,r + o(r)$. For $t > 0$, $W'(t) = \tilde{w}'(\sqrt{t})/(2\sqrt{t}) \to \tilde{w}''(0)/2$, and this limit is attained continuously, so $W \in C^1[0,T]$ with $W'(0) = \tilde{w}''(0)/2$. At the center all pure second derivatives of a radial C^2 function agree, so $\Delta w(0) = d\,\tilde{w}''(0) = 2d\,W'(0)$. The last claim is immediate since $\nu > 0$. ∎

Now let $u \in C^{2m}(\overline{B}_R)$ be radial with $\Delta^m u \geq 0$. Each $\Delta^k u$, $0 \leq k \leq m-1$, is radial of class $C^{2(m-k)} \subseteq C^2$, so Lemma 2.1 applies to it: writing $U_k = L^k U$ for the t-form of $\Delta^k u$, we obtain the *tower*

$$U_k \in C^1[0,T], \qquad U_k' \;=\; t^{-\nu} \cdot \big(t^\nu U_k'\big), \qquad \big(t^\nu U_k'\big)' \;=\; \tfrac{t^{\nu-1}}{4}\, U_{k+1} \quad \text{on } (0,T), \tag{2}$$

with the crucial automatic conditions

$$t^\nu\, U_k'(t) \;\longrightarrow\; 0 \qquad (t \to 0^+), \qquad 0 \leq k \leq m-1. \tag{3}$$

Lemma 2.2 (Kernel and rigidity)**.** *The radial m-polyharmonic functions of class $C^{2m}(\overline{B}_R)$ are exactly $\sum_{i=0}^{m-1} a_i\, |x|^{2i}$, i.e. the polynomials of degree $\leq m-1$ in t. On monomials, $L\, t^k = \lambda_k\, t^{k-1}$ with $\lambda_k = 2k\,(2k+d-2) > 0$, so with $\Lambda_k = \prod_{i=1}^k \lambda_i$ $(\Lambda_0 = 1)$,*

$$\Delta^k u(0) \;=\; \Lambda_k\, a_k \qquad (0 \leq k \leq m-1),$$

and the data map $p \mapsto (\gamma_0, \ldots, \gamma_{m-2}; \beta)$ is a linear isomorphism from the kernel onto $\mathbb{R}^m$. We write v_δ for the interpolant: $a_k = \gamma_k/\Lambda_k$ for $k \leq m-2$, the top coefficient fixed by β. (This is the smooth radial case of the Almansi expansion of polyharmonic functions; see [1].)

Proof. $\Delta |x|^{2k} = 2k(2k+d-2)\,|x|^{2k-2}$ gives the monomial formula, and iterating it the data formula. That the smooth radial kernel is no larger: a radial C^2 solution of $LQ = 0$ has $t^\nu Q' \equiv c$ by (1), and (3) forces $c = 0$, so Q is constant; induction on m then solves $LP = $ (kernel of L^{m-1}) within polynomials using $\lambda_k > 0$. Bijectivity: the matrix of the data map in the basis (t^k) is triangular with entries Λ_k on the diagonal, bordered by the row $(T^i)_i$; a kernel element with zero data has $a_k = 0$ for $k \leq m-2$ and then $a_{m-1} T^{m-1} = 0$. □

3. The greatest element and its consequences

We use the chain lemma of [4, Lemma 2.3] in a singular-endpoint version; the same two-line proof applies verbatim, since it uses only the sign of the derivative on the open interval together with continuity up to the endpoints. Say $h \in C[0,T]$ has *property (S)* if there is $\tau \in [0,T]$ with $h \leq 0$ on $[0,\tau)$ and $h \geq 0$ on $(\tau, T]$. Every nondecreasing function has (S); and if $H \in C[0,T] \cap C^1(0,T)$ satisfies $H' = w\, G$ on $(0,T)$ with $w > 0$ there and G having (S), then H is V-shaped (nonincreasing then nondecreasing), and if moreover $H(0) \leq 0$, then H has (S).

Theorem 3.1. *Let $m \geq 2$ and $\delta \in \mathbb{R}^m$.*

- (i) *$v_\delta \in \mathcal{K}(\delta)$, and $u \leq v_\delta$ on $\overline{B}_R$ for every $u \in \mathcal{K}(\delta)$. Consequently $\delta \in \mathcal{D}_m(R,d)$ if and only if the single radial polyharmonic function v_δ satisfies the maximum principle.*
- (ii) *If $\gamma_k \leq 0$ for $1 \leq k \leq m-2$, then every $u \in \mathcal{K}(\delta)$ is V-shaped in the radius; such δ is forcing for every β.*
- (iii) *$\mathcal{D}_m(R,d) = \mathcal{A} \cup \mathcal{B}$, where $\mathcal{A} = \{\delta : v_\delta \leq \gamma_0\}$ and $\mathcal{B} = \{\delta : v_\delta \leq \beta\}$ are closed convex cones.*

Proof. (i) Membership of v_δ is Lemma 2.2. Let $u \in \mathcal{K}(\delta)$ and put $h = U - v_\delta$ in the t-variable; write $h_k = L^k h$, so $h_k \in C^1[0,T]$ for $k \leq m-1$ by Lemma 2.1 (applied to $\Delta^k u$ and to the polynomial), $h_k(0) = 0$ for $0 \leq k \leq m-2$ (matching data), $h_0(T) = 0$, and by (3)

$$t^\nu h_k'(t)\big|_{t=0^+} \;=\; 0 \qquad (0 \leq k \leq m-1).$$

Top of the chain: on $(0,T)$, $\big(t^\nu h_{m-1}'\big)' = \frac{t^{\nu-1}}{4}\, L^m h = \frac{t^{\nu-1}}{4}\, \Delta^m u \geq 0$, so $t^\nu h_{m-1}'$ is nondecreasing on $(0,T]$ and tends to 0 at 0^+: hence $t^\nu h_{m-1}' \geq 0$, hence $h_{m-1}' \geq 0$ on $(0,T)$: h_{m-1} is nondecreasing, so it has (S).

Descent: fix k with $m-2 \geq k \geq 0$ and suppose h_{k+1} has (S). The function $G_k := t^\nu h_k'$ is continuous on $[0,T]$ (value 0 at 0), and $G_k' = \frac{t^{\nu-1}}{4} h_{k+1}$ on $(0,T)$: by the chain lemma G_k is

V-shaped, and since $G_k(0) = 0 \le 0$ it has (S). Then $h_k' = t^{-\nu} G_k$ has the same sign pattern on $(0, T)$, so h_k is V-shaped; for $k \ge 1$, $h_k(0) = 0$ upgrades this to (S), continuing the descent. At $k = 0$ we stop at the V-shape: $h = h_0$ is V-shaped with $h(0) = h(T) = 0$, so $h \le 0$.

The forcing criterion follows as always: members lie below v_δ, and v_δ is a member. (ii) is the same chain applied to U itself, the prescribed $\gamma_k \le 0$ replacing the zero data, ending in the V-shape of U, hence of u in the radius. (iii) is verbatim [3, Theorem 2.6]: the benchmark $\max\{\gamma_0, \beta\}$ equals one of its two arguments, and each pointwise condition is linear in δ. □

Corollary 3.2 (Biharmonic: all data force)**.** *Every radial $u \in C^4(\overline{B}_R)$ with $\Delta^2 u \ge 0$ is V-shaped in the radius; in particular it attains its maximum over $\overline{B}_R$ at the center or on the boundary, and $\mathcal{D}_2(R, d) = \mathbb{R}^2$.*

Proof. For $m = 2$ the condition set of Theorem 3.1(ii) is empty. □

Remark 3.3. Corollary 3.2 can also be assembled classically: $\Delta^2 u \ge 0$ makes Δu subharmonic, radial subharmonic functions are nondecreasing in the radius (their spherical means are themselves), and two integrations in (1) propagate the sign pattern. We record it because it is the $m = 2$ instance of a mechanism that continues uniformly to all orders, and because it marks where the forcing problem becomes nontrivial: at $m = 3$, not $m = 2$.

4. Triharmonic functions: the sharp constant is $4d/R^2$

Theorem 4.1. *Let $m = 3$. Then $\delta = (\gamma_0, \gamma_1; \beta) \in \mathcal{D}_3(R, d)$ if and only if*

$$\gamma_1 \;=\; \Delta u(0) \;\le\; \frac{4d}{R^2}\left(\beta - \gamma_0\right)^+.$$

Proof. By Theorem 3.1 we must decide the maximum principle on $[0, T]$ for the quadratic $q(t) = v_\delta = \gamma_0 + a_1 t + a_2 t^2$, where $a_1 = \gamma_1/\lambda_1 = \gamma_1/(2d)$ by Lemma 2.2. As in [3, Theorem 3.1], the principle fails iff $q'(0) > 0 > q'(T)$, and by the trapezoidal identity $q(T) - q(0) = \frac{T}{2}(q'(0) + q'(T))$ this happens iff $a_1 > \frac{2}{T}(\beta - \gamma_0)^+$. Substituting $a_1 = \gamma_1/(2d)$ and $T = R^2$ gives the criterion. □

Example 4.2. On the unit ball of $\mathbb{R}^3$: a radial u with $\Delta^3 u \ge 0$ obeys $u \le \max\{u(0), \max_{\partial B} u\}$ for *every* admissible u with its data iff $\Delta u(0) \le 12\,(u|_{\partial B} - u(0))^+$. The one-dimensional case $d = 1$, $R = 1$ gives the constant 4 for even functions on $[-1, 1]$ with sixth derivative ≥ 0—halving the interval, as it were, relative to the constant 2 of [3] on $[0, 1]$, in accordance with the R^{-2} scaling.

5. The inherited geometry, and the overshoot

Corollary 5.1 (The ball inherits the polynomial geometry)**.** *The map*

$$\delta \;\longmapsto\; q_\delta(s) \;=\; v_\delta(Ts), \qquad s \in [0, 1],$$

is a linear isomorphism of $\mathbb{R}^m$ onto the polynomials of degree $\le m - 1$, carrying $\mathcal{D}_m(R, d)$ onto the endpoint-maximum cone $\mathcal{M}_{m-1}$ of [3]. Consequently, for every R and d:

(i) *for $m = 4$, $\mathcal{D}_4(R, d)$ is described completely by* [3, Theorem 5.6]*, applied to the data*

$$c_0 = \gamma_0, \qquad c_1 = \frac{R^2\gamma_1}{2d}, \qquad c_2 = \frac{R^4\gamma_2}{4d(d+2)}, \qquad \beta = \beta$$

of q_δ (here $\Lambda_2 = 2d \cdot 4(d + 2)$, and $c_2 = q_\delta''(0)$);

(ii) *for $m \ge 4$, $\mathcal{D}_m(R, d)$ is not a finite union of convex polyhedra;*

(iii) *the boundary characterization of* [3, §6] *holds for $\partial\mathcal{D}_m(R, d)$ for $m \ge 3$, and the contact-type stratification, wall connectedness, and closure poset for $m \ge 4$, verbatim.*

Proof. By Theorem 3.1(i), δ is forcing iff v_δ has its maximum over $[0,T]$ at an endpoint, iff $q_\delta \in \mathcal{M}_{m-1}$; the map is bijective by Lemma 2.2. The listed consequences are [3, Theorems 5.6, 5.2, 6.2–6.4]. ☐

Finally, the extremal overshoot. Let $E_m(R,d)$ be the supremum of $\max_{\overline{B}_R} u - \max\{u(0),\beta\}$ over radial $u \in C^{2m}$ with $\Delta^m u \geq 0$ and $|\Delta^k u(0)| \leq 1$ for $1 \leq k \leq m-2$ (no constraint on $u(0)$ or β).

Theorem 5.2. *$E_m(R,d) = \max_{[0,T]} P$, where P is the kernel element with data $(0,1,\ldots,1;\,0)$:*

$$P(t) \;=\; \sum_{k=1}^{m-2} \frac{1}{\Lambda_k}\Big(t^k - T^k\big(t/T\big)^{m-1}\Big).$$

In particular

$$E_3(R,d) \;=\; \frac{R^2}{8\,d},$$

attained by $u = \frac{1}{2d}\big(|x|^2 - |x|^4/R^2\big)$.

Proof. As in [3, Theorem 6.8], in two steps, both inherited through Lemma 2.2. *Step 1:* $\partial v_\delta(t)/\partial\gamma_k = \big(t^k - T^k(t/T)^{m-1}\big)/\Lambda_k \geq 0$ on $[0,T]$, since $(t/T)^k \geq (t/T)^{m-1}$; the benchmark is unaffected, so the supremum saturates $\gamma_k = 1$. *Step 2:* the overshoot then depends on (γ_0,β) only through $\tau = \beta - \gamma_0$, via $O(\tau) = \varphi(\tau) - \tau^+$ with φ nondecreasing and $\varphi(\tau) - \tau$ nonincreasing (the coefficient of τ in $v_\delta(t)$ is $(t/T)^{m-1} \in [0,1]$); hence $O(\tau) \leq O(0) = \max P$ for all $\tau \in \mathbb{R}$, with equality at $\tau = 0$. For $m = 3$: $P(t) = \big(t - t^2/T\big)/(2d)$, maximized at $t = T/2$ with value $T/(8d)$; translating back, the extremal is $\frac{1}{2d}(|x|^2 - |x|^4/R^2)$, with $\Delta u(0) = 1$ and $u = 0$ at the center and boundary. ☐

Remark 5.3. $\frac{1}{2d}(|x|^2 - |x|^4/R^2)$ is the ball's incarnation of $x - x^2$, closing the same loop as in [3, 5]: the simplest function witnessing that $\Delta^3 u \geq 0$ alone does not force the maximum principle is also the extremal witness for how badly it can fail.

6. Problems

Problem 6.1. Drop radiality—with the right data. The naive extension of our setup fails outright: for $d \geq 2$ the functions $w_c(x) = c\,(R^2 - |x|^2)\,x_1$ satisfy $\Delta w_c = -2(d+2)\,c\,x_1$, hence $\Delta^2 w_c = 0$, and carry zero data ($w_c|_{\partial B_R} = 0$ and $\Delta^k w_c(0) = 0$ for every k), so the class with prescribed center data and boundary value alone is unbounded at interior points: no pointwise-greatest element exists. The natural repair is *Navier data*: prescribe $u, \Delta u, \ldots, \Delta^{m-1} u$ on $\partial\Omega$, for an arbitrary bounded domain Ω. The problem then factors into iterated Dirichlet Laplacians, and applying the second-order maximum principle m times shows that the class $\{\Delta^m u \geq 0\}$ with given Navier data has a pointwise-greatest element for m *odd* and a pointwise-least element for m *even*—the parity of [3, Remark 6.1] resurfacing at the PDE level. Develop the forcing theory: which Navier data force $u \leq \max_{\partial\Omega} u$? For $m = 3$ and constant data (c_0, c_1, c_2) the greatest element is $c_0 + c_1 T_\Omega + c_2 S_\Omega$, where $\Delta T_\Omega = 1$, $\Delta S_\Omega = T_\Omega$, and T_Ω, S_Ω vanish on $\partial\Omega$ (so $T_\Omega \leq 0 \leq S_\Omega$); the sharp constants thus become domain functionals such as $\sup_\Omega S_\Omega/(-T_\Omega)$, which on B_R equals $(d+4)\,R^2/\big(4d(d+2)\big)$—a new constant alongside those of this paper, as befits different data.

Problem 6.2. Reconcile ball and annulus quantitatively: [4, §5] treats radial polyharmonic forcing on annuli $\{r_0 \leq |x| \leq R\}$ with full quasi-derivative data at r_0. For families extending regularly across the center, the odd tower entries vanish as $r_0 \to 0^+$ by (3)—but not for singular branches ($u = \log r$ in $d = 2$ has $r u' \equiv 1$)—and the top even datum $\Delta^{m-1} u(r_0)$ is dropped as well, not merely the odd entries. Describe the convergence of the annular forcing sets to $\mathcal{D}_m(R,d)$ after restricting to regular families, setting the odd data to zero, and projecting out the top even datum.

Problem 6.3. Minimal regularity: the arguments used $u \in C^{2m}$ only through Lemma 2.1 applied to $\Delta^k u$, $k \leq m-1$. Formulate the theory for the natural wider class (distributional $\Delta^m u \geq 0$ with $\Delta^{m-1} u$ continuous), where the chain should survive with nondecreasing replaced by measure-theoretic monotonicity.

Problem 6.4. The constants $\lambda_k = 2k(2k+d-2)$ make every sharp constant explicit in d; for instance the triharmonic threshold $4d/R^2$ grows linearly in dimension. Determine the asymptotics of the full forcing region $\mathcal{D}_m(R,d)$ as $d \to \infty$ under natural normalizations, where the kernel monomials $|x|^{2k}$ concentrate near the boundary.

Acknowledgments

Fourth in a series descending from [2]: the forcing problem was solved for $f^{(n)} \geq 0$ in [3], freed from polynomials in [4], discretized in [5], and arrives here at polyharmonic operators.

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