

CENTRALITY OF THE BALANCED TWO-POINT OVERSHOOT

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ABSTRACT. Ninth in a series descending from [1]: we prove the centrality conjecture left open in [8, 9]. For the class $u^{(2m)} \geq 0$ with balanced two-point Hermite data, the worst-case overshoot under unit data bounds is $E_m = \max_{[0,1]} S_m$ for an explicit universal extremal S_m ; it was proved in [9] that $E_m \rightarrow \sqrt{e} - 1$, but the exact value of E_m at each finite order rested on the conjecture that the symmetric function S_m is maximized at the midpoint—verified through $m = 24$, unproven in general. We prove it, for every m , in three steps: the derivative of a binomial tail is a single Bernstein monomial; in S'_m the resulting symmetric terms s^{m-1} , $s = x(1-x)$, cancel exactly; and what remains dominates its own mirror monomial by monomial. Hence S_m increases strictly on $[0, \frac{1}{2}]$, the midpoint is the unique maximizer, and

$$E_m = S_m\left(\frac{1}{2}\right) \quad \text{exactly, for every } m.$$

The exact overshoot table of [8] is thereby unconditional at every order, and the sharp rate of approach to the ceiling follows:

$$E_m = \sqrt{e} - 1 - \frac{\sqrt{e}}{2} \binom{2m-2}{m-1} 4^{1-m} (1 + O(m^{-1})) = \sqrt{e} - 1 - \frac{\sqrt{e}}{2\sqrt{\pi m}} + O(m^{-3/2}),$$

settling the balanced case of [9, Problem 7.2]. The argument is genuinely balanced: the exact cancellation of the s^{m-1} terms is what symmetry buys, and the asymmetric maximizer problem remains open.

1. INTRODUCTION

For the class $u^{(2m)} \geq 0$ on $[0, 1]$ with prescribed $u^{(k)}(0), u^{(k)}(1)$ ($0 \leq k \leq m-1$), the largest possible excess of a member over $\max\{u(0), u(1)\}$, when the derivative data are bounded by one, equals $\max_{[0,1]} S_m$ for the single universal extremal

$$S_m(x) = \sum_{k=1}^{m-1} [H_{k0}(x) + H_{k0}(1-x)], \quad H_{k0}(x) = \frac{x^k}{k!} \mathbb{P}(\text{Bin}(2m-1-k, x) \leq m-1-k),$$

where H_{k0} are the (m, m) two-point Hermite cardinal polynomials, expressed as binomial tails as in [9, Lemma 4.2]. The function S_m is symmetric about $x = \frac{1}{2}$; the values $S_m(\frac{1}{2})$ are exactly computable and tend to $\sqrt{e} - 1$ [8, Theorem 7.2]; and [9, Theorem 6.2] proved $E_m \rightarrow \sqrt{e} - 1$ unconditionally. What remained open—[8, Problem 8.2], restated in [9, Problems 7.1–7.2]—was *centrality*: that the midpoint is where S_m attains its maximum, so that the exact finite-order values $E_m = S_m(\frac{1}{2})$ of [8, Theorem 7.1] hold as stated at every order, not just for the $m \leq 24$ verified in rational arithmetic.

This note proves centrality (Theorem 2.3) and, with it, the sharp rate of approach to the ceiling (Theorem 3.1). The proof is short because the binomial-tail representation of [9] turns the derivative S'_m into a sum with visible structure: differentiating a binomial tail produces a lone Bernstein monomial (Lemma 2.1); for each cardinal that monomial is a multiple of s^{m-1}

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with $s = x(1-x)$, *symmetric* in $x \leftrightarrow 1-x$, so it cancels from the derivative of the symmetrized sum (Lemma 2.2); and the surviving terms dominate their mirrors monomial by monomial on $[0, \frac{1}{2}]$ (proof of Theorem 2.3). Symmetry is used exactly once, in the cancellation—which is why the asymmetric maximizer problem of [9, Problem 7.1] is untouched by this argument.

2. CENTRALITY

Lemma 2.1 (Tail derivative). *For integers $0 \leq t < N$,*

$$\frac{d}{dx} \mathbb{P}(\text{Bin}(N, x) \leq t) = -N \binom{N-1}{t} x^t (1-x)^{N-1-t}.$$

Proof. Differentiating $\sum_{i \leq t} \binom{N}{i} x^i (1-x)^{N-i}$ termwise and using $i \binom{N}{i} = N \binom{N-1}{i-1}$ and $(N-i) \binom{N}{i} = N \binom{N-1}{i}$ gives

$$N \sum_{i=0}^t \left[\binom{N-1}{i-1} x^{i-1} (1-x)^{N-i} - \binom{N-1}{i} x^i (1-x)^{N-1-i} \right],$$

which telescopes to the $i = t$ negative term. $\square$

Lemma 2.2 (Cancellation). *Fix $m \geq 2$ and $1 \leq k \leq m-1$, and set*

$$G_k(x) := \frac{x^{k-1}}{(k-1)!} \mathbb{P}(\text{Bin}(2m-1-k, x) \leq m-1-k).$$

Then

$$\frac{d}{dx} [H_{k0}(x) + H_{k0}(1-x)] = G_k(x) - G_k(1-x);$$

in particular $S'_m(x) = \sum_{k=1}^{m-1} [G_k(x) - G_k(1-x)]$.

Proof. Write $N_k = 2m-1-k$ and $t_k = m-1-k$, so $H_{k0} = \frac{x^k}{k!} F_k$ with $F_k(x) = \mathbb{P}(\text{Bin}(N_k, x) \leq t_k)$. By the product rule and Lemma 2.1,

$$H'_{k0}(x) = G_k(x) - \frac{N_k}{k!} \binom{N_k-1}{t_k} x^{k+t_k} (1-x)^{N_k-1-t_k} = G_k(x) - a_k s^{m-1},$$

since $k + t_k = m-1$ and $N_k - 1 - t_k = m-1$, where $a_k = N_k \binom{N_k-1}{t_k} / k!$ and $s = x(1-x)$. The term $a_k s^{m-1}$ is symmetric under $x \mapsto 1-x$, so in

$$\frac{d}{dx} [H_{k0}(x) + H_{k0}(1-x)] = H'_{k0}(x) - H'_{k0}(1-x)$$

the two copies cancel, leaving $G_k(x) - G_k(1-x)$. $\square$

Theorem 2.3 (Centrality). *For every $m \geq 2$, S_m is strictly increasing on $[0, \frac{1}{2}]$ and, by symmetry, strictly decreasing on $[\frac{1}{2}, 1]$. Consequently the midpoint is the unique maximizer of S_m , and*

$$E_m = S_m\left(\frac{1}{2}\right) \quad \text{for every } m.$$

Proof. By Lemma 2.2 it suffices to show $G_k(x) \geq G_k(1-x)$ for $0 \leq x \leq \frac{1}{2}$, strictly for $0 < x < \frac{1}{2}$ when $k = 1$. Expanding the tail, with $N = 2m-1-k$,

$$(k-1)! G_k(x) = \sum_{i=0}^{m-1-k} \binom{N}{i} x^{k-1+i} (1-x)^{N-i},$$

and the mirror $G_k(1-x)$ has the same coefficients with the exponents exchanged. For each i in the range,

$$(N-i) - (k-1+i) = 2(m-k-i) \geq 2 > 0,$$

so every x -exponent is strictly smaller than its partner $(1-x)$ -exponent. For $0 < x < \frac{1}{2}$ and $a < b$ we have $x^a(1-x)^b > x^b(1-x)^a$, since the ratio is $(\frac{1-x}{x})^{b-a} > 1$; at $x \in \{0, \frac{1}{2}\}$ the comparison is an equality or trivial. Summing over i and k gives $S'_m \geq 0$ on $[0, \frac{1}{2}]$, and $S'_m > 0$ on $(0, \frac{1}{2})$ because the $k=1, i=0$ term is strictly positive there. The final claim is [8, Theorem 7.1]: $E_m = \max S_m$, now attained at $x = \frac{1}{2}$ alone. $\square$

Corollary 2.4. *The exact overshoot values of [8, Theorem 7.1] hold unconditionally at every order: $E_2 = \frac{1}{4}$, $E_3 = \frac{11}{32}$, $E_4 = \frac{151}{384}$, $E_5 = \frac{2609}{6144}$, $E_6 = \frac{54901}{122880}$, and in general $E_m = S_m(\frac{1}{2})$, an explicit rational number. The parenthetical conditions in [9, Corollary 6.3 and Remark 6.4] are discharged: the balanced maximizer is pinned at the midpoint for every order, not conjecturally.*

3. THE SHARP RATE

Theorem 3.1 (Deficit). *Let $A_m = \binom{2m-2}{m-1} 4^{1-m}$. Then*

$$\sqrt{e} - 1 - E_m = \frac{\sqrt{e}}{2} A_m (1 + O(m^{-1})) = \frac{\sqrt{e}}{2\sqrt{\pi m}} + O(m^{-3/2}).$$

Equivalently, in the order $n = 2m$: $E = \sqrt{e} - 1 - \sqrt{e/(2\pi n)} + O(n^{-3/2})$.

Proof. By Theorem 2.3, $E_m = S_m(\frac{1}{2})$, and by the computation in [8, Theorem 7.2],

$$S_m(\tfrac{1}{2}) = \sum_{k=1}^{m-1} \frac{2^{-k}}{k!} (1 - 2^{1-m} M_{m,k}), \quad M_{m,k} = \sum_{j=m-k}^{m-1} \binom{m-1+j}{j} 2^{-j},$$

so that, with $\tau := \binom{2m-2}{m-1} 2^{-(m-1)}$ (the $j = m-1$ term),

$$\sqrt{e} - 1 - E_m = \sum_{k \geq m} \frac{2^{-k}}{k!} + 2^{1-m} \sum_{k=1}^{m-1} \frac{2^{-k}}{k!} M_{m,k}.$$

The first sum is $O(2^{-m}/m!)$, negligible. For the second, write the terms of $M_{m,k}$ as $\text{term}_j = \binom{m-1+j}{j} 2^{-j}$ and compare with the top one, $\tau = \text{term}_{m-1}$: the ratio of consecutive terms is

$$\frac{\text{term}_{j-1}}{\text{term}_j} = \frac{2j}{m-1+j} = 1 - \frac{m-1-j}{m-1+j} \in \left[1 - \frac{\ell}{m}, 1\right] \quad (j = m-\ell, 1 \leq \ell \leq m),$$

using $(m(\ell-1) \leq \ell(2m-1-\ell) \iff 0 \leq (\ell+1)(m-\ell))$. So the i -th term below the top satisfies $\tau \geq \text{term}_{m-1-i} \geq \tau \prod_{\ell=1}^i (1 - \frac{\ell}{m}) \geq \tau(1 - \frac{i^2}{m})$, whence

$$0 \leq k\tau - M_{m,k} \leq k\tau \cdot \min\left(1, \frac{k^2}{m}\right) \leq \frac{k^3\tau}{m}.$$

Summing against the weights $2^{-k}/k!$ and using $\sum_k k 2^{-k}/k! = \frac{1}{2}e^{1/2}$ and $\sum_k k^3 2^{-k}/k! < \infty$,

$$2^{1-m} \sum_{k=1}^{m-1} \frac{2^{-k}}{k!} M_{m,k} = 2^{1-m} \tau \left(\frac{\sqrt{e}}{2} + O(m^{-1}) \right) = \frac{\sqrt{e}}{2} A_m (1 + O(m^{-1})),$$

since $2^{1-m}\tau = A_m$. Finally $A_m = (\pi(m-1))^{-1/2}(1 + O(m^{-1}))$ by Stirling's formula, giving the displayed forms. $\square$

Remark 3.2. The argument for Theorem 2.3 is genuinely balanced. For an asymmetric split (p, q) the tail-derivative monomials are multiples of $x^{p-1}(1-x)^{q-1}$, which is not symmetric, so nothing cancels in the derivative of the extremal $S^{(p,q)}$; and indeed the maximizer is not at any symmetric point but drifts to the transition $\lambda = p/n$ on the moderate-deviation scale conjectured in [9, Problems 7.1–7.2], which remain open. Within the balanced case, what we

prove is monotonicity of S_m on the half-interval; the stronger conjecture of [8, Problem 8.2]—that the coefficients of S_m as a polynomial in $s = x(1 - x)$ are all nonnegative, verified there through $m = 24$ —remains open, though nothing in the series now depends on it (Problem 3.3).

Problem 3.3. Absolute monotonicity in s : show that S_m , written as a polynomial in $s = x(1 - x)$, has nonnegative coefficients for every m . Centrality (Theorem 2.3) is the first-derivative consequence; the full claim would give, for instance, monotonicity of E_m along the family in a strong termwise sense.

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